

Optimal error estimates of mixed FEMs for second order hyperbolic integro-differential equations with minimal smoothness on initial data

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Abstract

In this article, mixed finite element methods are discussed for a class of hyperbolic integro-differential equations (HIDEs). Based on a modification of the nonstandard energy formulation of Baker, both semidiscrete and completely discrete implicit schemes for an extended mixed method are analyzed and optimal $L^\infty(L^2)$ -error estimates are derived under minimal smoothness assumptions on the initial data. Further, quasi-optimal estimates are shown to hold in $L^\infty(L^\infty)$ -norm. Finally, the analysis is extended to the standard mixed method for HIDEs and optimal error estimates in $L^\infty(L^2)$ -norm are derived again under minimal smoothness on initial data.

Key Words. Hyperbolic integro-differential equation, mixed finite element method, semidiscrete Galerkin approximation, completely discrete implicit method, optimal error estimates, minimal smoothness on initial data.

1 Introduction

We consider two mixed finite element methods for the following hyperbolic integro-differential equation:

$$(1.1) \quad u_{tt} - \nabla \cdot \left(A \nabla u - \int_0^t B(t, s) \nabla u(s) ds \right) = 0 \quad \text{in } \Omega \times J,$$

$$(1.2) \quad u(x, t) = 0 \quad \text{on } \partial\Omega \times J,$$

$$(1.3) \quad u(x, 0) = u_0 \quad \text{in } \Omega,$$

$$(1.4) \quad u_t(x, 0) = u_1 \quad \text{in } \Omega,$$

with given functions u_0 and u_1 , where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain, $J = (0, T]$, $T < \infty$, and $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. Here, $A = [a_{ij}(x)]_{1 \leq i, j \leq d}$ and $B(t, s) = [b_{ij}(x, t, s)]_{1 \leq i, j \leq d}$ are 2×2 matrices with smooth coefficients. Further, we assume that A is symmetric and uniformly positive definite in $\bar{\Omega}$. Problems of this kind arise in linear viscoelasticity specially in viscoelastic materials with memory (cf. Renardy *et al.* [18]).

Early *a priori* error estimates for Galerkin finite element methods for solving (1.1)-(1.4) (without integral terms) were derived by Dupont [10] using a standard energy argument. These estimates were improved by Baker [2], who used a technique interpreted later in literature as a nonstandard energy arguments. In [17], Rauch discussed the convergence of a continuous time Galerkin approximation to a second order wave equation and proved optimal error estimates in $L^\infty(L^2)$ -norm using piecewise linear polynomials, when $u_0 \in H^3 \cap H_0^1$ (with $u_1 = 0$). The analysis improves upon the earlier results of Baker and Dougalis [3], where optimal error estimates were derived under the assumptions that $u_0 \in H^5 \cap H_0^1$ and $u_1 \in H^4 \cap H_0^1$. In [19], Sinha and Pani extended Rauch's results to hyperbolic integro-differential equation with quadrature and obtained optimal error estimates in $L^\infty(L^2)$ norm in the case $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. For more on Galerkin methods and optimal error estimates for the problem (1.1)-(1.4), see, [6, 16].

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In the literature, optimal error estimates for mixed finite element approximations to second order hyperbolic equations were proved in [9, 13, 15, 4, 8]. In [13], Geveci derived L^∞ -in-time, L^2 -in-space error bounds for the continuous-in-time mixed finite element approximations of velocity and stress. In [9], *a priori* error estimates were obtained for mixed finite element approximations to both displacement and stress requiring less regularity than needed in [13]. Stability for a family of discrete-in-time schemes was also demonstrated under regularity $u_0 \in H^4 \cap H_0^1$ and $u_1 \in H^3$, when $r = 2$. In [15], a mixed finite element displacement formulation was proposed for the acoustic wave equation under reduced regularity requirement on the displacement variable. Further, for a completely discrete explicit scheme, stability results and error estimates were established.

For the standard mixed methods, the problem (1.1) is often rewritten by introducing a new variable

$$(1.5) \quad \boldsymbol{\sigma}(t) = A \nabla u - \int_0^t B(t, s) \nabla u(s) ds,$$

or equivalently

$$(1.6) \quad \alpha \boldsymbol{\sigma}(t) = \nabla u - \int_0^t A^{-1} B(t, s) \nabla u(s) ds,$$

as

$$(1.7) \quad u_{tt} - \nabla \cdot \boldsymbol{\sigma}(t) = 0,$$

where $\alpha = A^{-1}$. Further, using resolvent operator, (1.6) is formulated as

$$(1.8) \quad \nabla u(t) = \alpha \boldsymbol{\sigma}(t) + \int_0^t M(t, s) \boldsymbol{\sigma}(s) ds,$$

where $M(t, s) = R(t, s) A^{-1}$ and $R(t, s)$ is the resolvent of the matrix $A^{-1} B(t, s)$ given by

$$(1.9) \quad R(t, s) = A^{-1} B(t, s) + \int_s^t A^{-1} B(t, \tau) R(\tau, s) d\tau, \quad t > s \geq 0.$$

With $W = L^2(\Omega)$ and $\mathbf{V} = \mathbf{H}(\text{div}; \Omega)$, the weak formulation for the mixed problem (1.7)-(1.8) is to seek a pair of functions $(u, \boldsymbol{\sigma}) : (0, T] \rightarrow W \times \mathbf{V}$ satisfying

$$(1.10) \quad (\alpha \boldsymbol{\sigma}, \mathbf{v}) + \int_0^t (M(t, s) \boldsymbol{\sigma}(s), \mathbf{v}) ds + (\nabla \cdot \mathbf{v}, u) = 0, \quad \mathbf{v} \in \mathbf{V},$$

$$(1.11) \quad (u_{tt}, w) - (\nabla \cdot \boldsymbol{\sigma}, w) = 0, \quad w \in W$$

with $u(0) = u_0$ and $u_t(0) = u_1$. To the best of our knowledge, there are few results available on optimal estimates of mixed approximations to problem (1.1)-(1.4) with minimum smoothness on initial data. In the context of parabolic problems, based on mixed methods related to the weak formulation (1.10)-(1.11), Sinha *et al.* [21] have derived an optimal convergence rate $O(ht^{-1})$ for the velocity $\boldsymbol{\sigma}(t)$ in \mathbf{L}^2 -norm and suboptimal convergence rate $O(ht^{-1/2})$ for the pressure $u(t)$ in L^2 -norm with $t \in (0, T]$, when $u_0 \in L^2(\Omega)$. However, optimal rate $O(h^2 t^{-1})$ for u is only established for a class of problems when $A = aI$ and $B = b(t, s)I$, where a and b are independent of spatial variable x . These results have recently been improved by Goswami *et al.* [14], who have established optimal convergence rate $O(h^2 t^{-1})$ for $u(t)$ in L^2 -norm and $O(ht^{-1})$ for $\boldsymbol{\sigma}(t)$ \mathbf{L}^2 -norm for all $t \in (0, T]$, when $u_0 \in L^2(\Omega)$. Similar results have been obtained for the extended finite element method described below. In both papers only semidiscrete problems have been considered.

In the first part of this paper, an extended mixed method for (1.1) is proposed and analyzed. Such a method was analysed earlier in [7] and [1] for elliptic problem, [23] for degenerate nonlinear parabolic problem, [14] for parabolic integro-differential equations and references cited in. To motivate this new mixed method, we introduce two variables:

$$(1.12) \quad \mathbf{q} = \nabla u, \quad \text{and} \quad \boldsymbol{\sigma} = A \mathbf{q} - \int_0^t B(t, s) \mathbf{q}(s) ds.$$

Then, the equation (1.1) takes the form

$$u_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0.$$

Now, the weak mixed formulation of (1.1)-(1.4) which forms a basis of our mixed Galerkin method is to find $(u, \mathbf{q}, \boldsymbol{\sigma}) : J \rightarrow W \times \mathbf{V} \times \mathbf{V}$ satisfying

$$(1.13) \quad (\mathbf{q}, \mathbf{v}) + (u, \nabla \cdot \mathbf{v}) = 0 \quad \mathbf{v} \in \mathbf{V},$$

$$(1.14) \quad (\boldsymbol{\sigma}, \mathbf{z}) - (A\mathbf{q}, \mathbf{z}) + \int_0^t (B(t, s)\mathbf{q}(s), \mathbf{z})ds = 0 \quad \mathbf{z} \in \mathbf{V},$$

$$(1.15) \quad (u_{tt}, w) - (\nabla \cdot \boldsymbol{\sigma}, w) = 0 \quad w \in W,$$

with $u(0) = u_0$ and $u_t(0) = u_1$. The main goal of this paper is to establish optimal convergence rate for the approximate solutions of (1.1), when the initial functions $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. Our analysis is essentially based on a simple energy technique and a use of a time integration without exploiting the inverse of the associated discrete elliptic operator. Essentially, integrating in time leads to a first order evolution process and hence, is instrumental in reducing the regularity requirements on the solution. Further, due to the presence of the integral term, it is observed that the concept of mixed Ritz-Volterra projections used earlier in [11],[12],[21] and [14] plays a crucial role in our analysis. For the completely discrete scheme, a major difficulty associated with a use of the nonstandard energy formulation by Baker is the presence of a fourth order time derivative of the displacement u in the bounds, see [9]. Moreover, additional difficulty is caused by the quadrature error associated with a second order midpoint quadrature rule which is used to approximate the integral term. Special care is needed to arrest these issues otherwise for optimal convergence rate, we land up with higher regularity assumption on the initial data, namely; $u_0 \in H^4 \cap H_0^1$. Therefore, a modification of Baker's approach is adopted and optimal error estimates in $\ell^\infty(L^2)$ -norm are derived, when $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. This technique is proved to be powerful and can successfully be applied even to the problem in [19]. Finally, the analysis has been extended to the standard mixed method corresponding to the formulation (1.10)-(1.11) and error analysis has been briefly discussed.

Now compared to the (1.10)-(1.11), the extended or expanded method corresponding to (1.13)-(1.15) may have introduced one more variable leading to a computation of one more extra variable. However, in Section 2, it is shown that it is possible to eliminate \mathbf{q}_h , which is an approximation of the gradient vector \mathbf{q} . Hence, both these schemes have almost comparable computational cost. Moreover, in the new formulation, we need not invert the coefficient matrix A .

Throughout this article, we denote by C , a generic positive constant which may vary from context to context, and whenever there arises no confusion, we would denote $u(t)$ simply as u for the sake of convenience.

An outline of the paper is as follows. In Section 2, we give some *a priori* bounds and regularity results for (1.13)-(1.15), and briefly present the finite element approximation of the extended mixed formulation (1.13)-(1.15). In Section 3, the extended mixed Ritz-Volterra projection is introduced and analyzed. In Section 4, error estimates for Galerkin approximations of $u, \boldsymbol{\sigma}$ and \mathbf{q} for the semidiscrete problem are derived. The completely discrete problem is discussed in Section 5 and optimal error estimates in $\ell^\infty(L^2)$ -norm are established. Finally, in Section 6, an extension of our analysis to the mixed formulation (1.10)-(1.11) with minimal smoothness assumptions on the initial data is briefly discussed.

2 Extended Mixed Finite Element Method

For our analysis, we shall use the standard notations for $L^2(\Omega)$, $H_0^1(\Omega)$, $H^m(\Omega)$ and $\mathbf{H}(\text{div}; \Omega)$ spaces with their norms and seminorms. To be more specific, $L^2(\Omega)$ is equipped with inner product (\cdot, \cdot) and norm $\|\cdot\|$. The norm on $\mathbf{H}(\text{div}; \Omega)$ is given by

$$\|\mathbf{v}\|_{\mathbf{H}(\text{div}; \Omega)} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2}.$$

Further, the standard Sobolev space $H^m(\Omega)$ of order m is equipped with its norm $\|\cdot\|_{H^m(\Omega)}$, which we denote simply by $\|\cdot\|_m$. Since the matrix A in (1.1) is positive definite, there exist positive constants a_0 and a_1 such that

$$(2.1) \quad a_0 \|\boldsymbol{\sigma}\| \leq \|\boldsymbol{\sigma}\|_A \leq a_1 \|\boldsymbol{\sigma}\|, \quad \text{where } \|\boldsymbol{\sigma}\|_A^2 := (A\boldsymbol{\sigma}, \boldsymbol{\sigma}).$$

Further, assume that all coefficients of B and their derivatives are bounded in their respective domain of definitions by the positive constant a_1 . Under our assumptions on the domain and on the coefficient matrix A , we note that the following elliptic regularity result holds: there exists a positive constant C such that for $\phi \in H^2 \cap H_0^1$

$$(2.2) \quad \|\phi\|_2 \leq C \|\nabla \cdot (A \nabla \phi)\|.$$

For our subsequent use, we state without proof *a priori* estimates for u , \mathbf{q} and $\boldsymbol{\sigma}$ satisfying (1.13)-(1.15) under appropriate regularity conditions on the initial data u_0 and u_1 . For more details, we refer to [19] and [20].

Lemma 2.1 *Let $(u, \mathbf{q}, \boldsymbol{\sigma})$ satisfy (1.13)-(1.15). Then, there is a positive constant C such that the following regularity results hold:*

$$\|D_t^j u(t)\| + \|D_t^{j-1} u(t)\|_1 + \|D_t^{j-1} \boldsymbol{\sigma}(t)\| + \|D_t^{j-1} \mathbf{q}(t)\| \leq C(T)(\|u_0\|_j + \|u_1\|_{j-1}), \quad j = 1, \dots, 4,$$

and

$$\|D_t^j u(t)\|_2 \leq C(T)(\|u_0\|_{j+2} + \|u_1\|_{j+1}), \quad j = 0, 1, 2,$$

where $D_t^j = (\partial^j / \partial t^j)$.

Based on the mixed formulation (1.13)-(1.15) for the problem (1.1)-(1.3), we now introduce the extended mixed finite element Galerkin method. Let \mathcal{T}_h be a regular triangulation of Ω by triangles of diameter at most h . Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying the following conditions:

- (i) $\nabla \cdot \mathbf{V}_h \subset W_h$, and
- (ii) there exists a linear operator $\Pi_h : \mathbf{V} \rightarrow \mathbf{V}_h$ such that $\nabla \cdot \Pi_h = P_h(\nabla \cdot)$,

where $P_h : W \rightarrow W_h$ is the L^2 -projection defined by

$$(\phi - P_h \phi, w_h) = 0, \quad \forall w_h \in W_h, \quad \phi \in W.$$

Further, assume that the finite element spaces satisfy the following approximation properties:

$$(2.3) \quad \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\| \leq Ch^r \|\nabla \cdot \boldsymbol{\sigma}\|_{r-1}, \quad \|u - P_h u\| \leq Ch^r \|u\|_r, \quad r = 1, 2,$$

and on a quasi-uniform mesh,

$$(2.4) \quad \|u - P_h u\|_{L^\infty(\Omega)} \leq Ch^r |\log h|^{1/2} \|u\|_{r+1}, \quad r = 1, 2.$$

Although, we can have several choices for \mathbf{V}_h and W_h , here we consider only the Raviart-Thomas elements of order one [5]. Note that P_h and Π_h satisfy

$$(2.5) \quad (\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}), w_h) = 0, \quad w_h \in W_h; \quad (u - P_h u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h.$$

Now, the corresponding semidiscrete mixed finite element formulation is to seek a triplet $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h) : (0, T] \rightarrow W_h \times \mathbf{V}_h \times \mathbf{V}_h$ satisfying

$$(2.6) \quad (\mathbf{q}_h, \mathbf{v}_h) + (u_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$$(2.7) \quad (\boldsymbol{\sigma}_h, \mathbf{z}_h) - (A \mathbf{q}_h, \mathbf{z}_h) + \int_0^t (B(t, s) \mathbf{q}_h(s), \mathbf{z}_h) ds = 0 \quad \forall \mathbf{z}_h \in \mathbf{V}_h,$$

$$(2.8) \quad (u_{htt}, w_h) - (\nabla \cdot \boldsymbol{\sigma}_h, w_h) = 0 \quad \forall w_h \in W_h,$$

with initial data $u_h(0)$ and $u_{ht}(0)$ to be defined later. Since W_h and \mathbf{V}_h are finite dimensional spaces, the discrete problem (2.6)-(2.8) leads to a linear system consisting of differential, integral and algebraic equations. Let $\{v_i\}_{i=1}^{N_1}$ and $\{\psi_i\}_{i=1}^{N_2}$ be the basis functions of the finite element spaces W_h and \mathbf{V}_h , respectively. Let

$$u_h(t) = \sum_{i=1}^{N_1} \alpha_i(t) v_i(x), \quad \mathbf{q}_h(t) = \sum_{i=1}^{N_2} \beta_i(t) \psi_i(x), \quad \boldsymbol{\sigma}_h(t) = \sum_{i=1}^{N_2} \gamma_i(t) \psi_i(x),$$

and $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_{N_1}(t))^T$, $\beta(t) = (\beta_1(t), \beta_2(t), \dots, \beta_{N_2}(t))^T$, and $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_{N_2}(t))^T$. By choosing the test functions $\mathbf{v}_h = \psi_j(x)$ and $\mathbf{z}_h = \psi_j(x)$, for $j = 1, 2, \dots, N_2$, in (2.6) and (2.7), respectively, and $w_h = v_j(x)$ in (2.8), we obtain the following system:

$$(2.9) \quad \mathbb{D}\beta(t) + \mathbb{C}\alpha(t) = 0,$$

$$(2.10) \quad \mathbb{D}\gamma(t) - \mathbb{A}\beta(t) + \int_0^t \mathbb{B}(t, s)\beta(s) ds = 0,$$

$$(2.11) \quad \mathbb{D}\alpha''(t) - \mathbb{C}^T\gamma(t) = 0,$$

with $\alpha(0)$, $\alpha'(0)$, $\beta(0)$, and $\gamma(0)$ are given from the initial data of the system (2.6)-(2.8). The matrices in (2.9)-(2.11) are defined as follows

$$\mathbb{D} = [(\psi_i, \psi_j)]_{N_2 \times N_2}, \quad \mathbb{A} = [(A\psi_i, \psi_j)]_{N_2 \times N_2},$$

$$\mathbb{B}(t, s) = [(B(t, s)\psi_i, \psi_j)]_{N_2 \times N_2}, \quad \mathbb{C} = [(v_i, \operatorname{div}\psi_j)]_{N_1 \times N_2}.$$

From (2.9), we obtain $\beta(t) = -\mathbb{D}^{-1}\mathbb{C}\alpha(t)$. After elimination of $\beta(t)$, this can be seen a system of integro-differential equation

$$(2.12) \quad \mathbb{D}\alpha''(t) - \mathbb{C}^T\gamma(t) = 0,$$

$$(2.13) \quad \mathbb{D}\gamma(t) + \mathbb{A}\mathbb{D}^{-1}\mathbb{C}\alpha(t) = \int_0^t \mathbb{B}(t, s)\mathbb{D}^{-1}\mathbb{C}\alpha(s) ds = 0.$$

Using Picard's method, it is easy to check that the system (2.13)-(2.12) has a unique solution.

Compared to the two field formulation which is based on the mixed weak form (1.10)-(1.11) and is stated in Section 6, the new mixed system (2.6)-(2.8) which depends on three field formulation has one more additional variable to compute. However, one variable, say \mathbf{q}_h can be easily eliminated with negligible computational cost and hence, it can have comparable computational cost. Further, there is no need of inverting the coefficient matrix A .

3 Mixed Ritz-Volterra Type Projections

In this section, the extended mixed Ritz-Volterra projections are introduced and analyzed. The projections are defined as follows: Given $(u(t), \mathbf{q}(t), \boldsymbol{\sigma}(t)) \in W \times \mathbf{V} \times \mathbf{V}$, for $t \in (0, T]$, find $(\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h) : (0, T] \rightarrow W_h \times \mathbf{V}_h \times \mathbf{V}_h$ satisfying

$$(3.1) \quad (\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{v}_h) + (\eta_u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(3.2) \quad (\boldsymbol{\eta}_{\boldsymbol{\sigma}}, \mathbf{z}_h) - (A\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{z}_h) + \int_0^t (B(t, s)\boldsymbol{\eta}_{\mathbf{q}}(s), \mathbf{z}_h) ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(3.3) \quad (\nabla \cdot \boldsymbol{\eta}_{\boldsymbol{\sigma}}, w_h) = 0, \quad w_h \in W_h,$$

where $\eta_u = (u - \tilde{u}_h)$, $\boldsymbol{\eta}_{\mathbf{q}} = (\mathbf{q} - \tilde{\mathbf{q}}_h)$ and $\boldsymbol{\eta}_{\boldsymbol{\sigma}} = (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)$. Since W_h and \mathbf{V}_h are finite dimensional spaces, the discrete problem (3.1)-(3.3), for a given triplet $\{u, \mathbf{q}, \boldsymbol{\sigma}\}$, leads to a system of linear equations combined with algebraic constraints for $\{\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h\}$. Note that when $B = 0$, the system has a unique solution, see [7]. Now using theory of linear Volterra equations of second kind and Picard's iteration, it is straightforward to prove that the system (3.1)-(3.3), for a given triplet $\{u, \mathbf{q}, \boldsymbol{\sigma}\}$, has a unique solution $\{\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h\}$.

In this section, we discuss estimates of η_u , $\boldsymbol{\eta}_{\mathbf{q}}$ and $\boldsymbol{\eta}_{\boldsymbol{\sigma}}$. Using definitions of P_h and Π_h , we rewrite η_u , $\boldsymbol{\eta}_{\mathbf{q}}$ and $\boldsymbol{\eta}_{\boldsymbol{\sigma}}$ as

$$\begin{aligned} \boldsymbol{\eta}_{\mathbf{q}} &= (\mathbf{q} - P_h\mathbf{q}) - (\tilde{\mathbf{q}}_h - P_h\mathbf{q}) =: \boldsymbol{\theta}_{\mathbf{q}} - \boldsymbol{\rho}_{\mathbf{q}}, \\ \eta_u &= (u - P_h u) - (\tilde{u}_h - P_h u) =: \theta_u - \rho_u, \\ \boldsymbol{\eta}_{\boldsymbol{\sigma}} &= \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma} =: \boldsymbol{\theta}_{\boldsymbol{\sigma}}. \end{aligned}$$

Because the estimates of θ_u , $\theta_{\mathbf{q}}$ and θ_{σ} are known, it is sufficient to estimate ρ_u , $\rho_{\mathbf{q}}$. Now rewrite (3.1)-(3.3) as

$$(3.4) \quad (\rho_{\mathbf{q}}, \mathbf{v}_h) + (\rho_u, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$$(3.5) \quad - (A\rho_{\mathbf{q}}, \mathbf{z}_h) + \int_0^t (B(t, s)\rho_{\mathbf{q}}(s), \mathbf{z}_h) ds = (\theta_{\sigma}, \mathbf{z}_h) - (A\theta_{\mathbf{q}}, \mathbf{z}_h) \\ + \int_0^t (B(t, s)\theta_{\mathbf{q}}(s), \mathbf{z}_h) ds \quad \forall \mathbf{z}_h \in \mathbf{V}_h,$$

$$(3.6) \quad (\nabla \cdot \theta_{\sigma}, w_h) = 0 \quad \forall w_h \in W_h.$$

Below, we derive estimates of η_{σ} and $\eta_{\mathbf{q}}$. The analysis of Section 4 of [14] can be suitably modified to prove the following results in Lemmas 3.1 and 3.2, but for completeness, we indicate the proofs here.

Lemma 3.1 *Let $(\eta_u, \eta_{\mathbf{q}}, \eta_{\sigma})$ be such that the system (3.1)-(3.3) is satisfied. Then, there exists a constant C independent of h such that for $t \in (0, T]$*

$$(3.7) \quad \|D_t^j \eta_{\sigma}(t)\| + \|D_t^j \eta_{\mathbf{q}}(t)\| \leq Ch^r (\|u_0\|_{j+r+1} + \|u_1\|_{j+r}), \quad j = 0, 1, 2, \quad r = 1, 2,$$

and

$$(3.8) \quad \|\eta_{\mathbf{q}}(t)\|_{(\mathbf{H}(\text{div}; \Omega))^*} \leq Ch^2 (\|u_0\|_2 + \|u_1\|_1) + \|\eta_u\|,$$

where $(\mathbf{H}(\text{div}; \Omega))^*$ is the dual of $\mathbf{H}(\text{div}; \Omega)$.

Proof. First, observe that

$$\|D_t^j \eta_{\sigma}\| \leq Ch^r \|\nabla \cdot D_t^j \sigma\|_{r-1} \leq Ch^r \|D_t^j u\|_{r+1}, \quad j = 0, 1, 2, \quad r = 1, 2,$$

and

$$\|\nabla \cdot D_t^j \eta_{\sigma}\| \leq Ch^r \|\nabla \cdot D_t^j \sigma\|_r \leq Ch^r \|D_t^j u\|_{r+2}.$$

Next, choose $\mathbf{z}_h = \rho_{\mathbf{q}}$ in (3.5) to obtain

$$\|A^{1/2} \rho_{\mathbf{q}}\|^2 = -(\theta_{\sigma}, \rho_{\mathbf{q}}) + (A\theta_{\mathbf{q}}, \rho_{\mathbf{q}}) + \int_0^t (B(t, s)\theta_{\mathbf{q}}(s), \rho_{\mathbf{q}}) ds \\ - \int_0^t (B(t, s)\rho_{\mathbf{q}}(s), \rho_{\mathbf{q}}(t)) ds.$$

Then, a use of the Cauchy-Schwarz inequality with the boundedness of B and the positive definiteness property of A yields

$$(3.9) \quad \|\rho_{\mathbf{q}}\| \leq C(T, a_1) \left(\|\theta_{\sigma}\| + \|\theta_{\mathbf{q}}\| + \int_0^t (\|\theta_{\mathbf{q}}(s)\| + \|\rho_{\mathbf{q}}(s)\|) ds \right).$$

Notice that by (1.12), it follows that

$$(3.10) \quad \|\theta_{\mathbf{q}}\| \leq Ch^r \|\mathbf{q}\|_r \leq Ch^r \|u\|_{r+1}, \quad \text{and} \quad \|\theta_{\sigma}\| \leq Ch^r \|\nabla \cdot \sigma\|_{r-1} \leq Ch^r \|u\|_{r+1}.$$

A substitution of (3.10) in (3.9) with Lemma 2.1 shows that

$$\|\rho_{\mathbf{q}}\| \leq Ch^r (\|u_0\|_{r+1} + \|u_1\|_r) + C \int_0^t \|\rho_{\mathbf{q}}(s)\| ds.$$

An application of Gronwall's Lemma yields

$$\|\rho_{\mathbf{q}}\| \leq Ch^r (\|u_0\|_{r+1} + \|u_1\|_r).$$

A use of the triangle inequality establishes the estimate (3.7) for $j = 0$. Now, differentiate (3.5) with respect to time to obtain

$$\begin{aligned} (A\rho_{\mathbf{q}_t}, \mathbf{z}_h) &= -(\boldsymbol{\theta}_{\sigma t}, \mathbf{z}_h) + (A\boldsymbol{\theta}_{\mathbf{q}_t}, \mathbf{z}_h) - (B(t, t)\boldsymbol{\theta}_{\mathbf{q}}(t), \mathbf{z}_h) \\ &\quad - \int_0^t (B_t(t, s)\boldsymbol{\theta}_{\mathbf{q}}(s), \mathbf{z}_h) ds + (B(t, t)\rho_{\mathbf{q}}(t), \mathbf{z}_h) + \int_0^t (B_t(t, s)\rho_{\mathbf{q}}(s), \mathbf{z}_h) ds. \end{aligned}$$

Again, apply the Cauchy-Schwarz inequality and the boundedness of A and B to arrive at

$$\|\rho_{\mathbf{q}_t}\| \leq C(T, a_1) \left(\|\rho_{\mathbf{q}}\| + \|\boldsymbol{\theta}_{\mathbf{q}}\| + \|\boldsymbol{\theta}_{\mathbf{q}_t}\| + \|\boldsymbol{\theta}_{\sigma t}\| + \int_0^t (\|\boldsymbol{\theta}_{\mathbf{q}}(s)\| + \|\rho_{\mathbf{q}}(s)\|) ds \right).$$

Taking into account approximation properties (2.3) and (1.12), it follows that

$$\|\boldsymbol{\theta}_{\mathbf{q}_t}\| \leq Ch^r \|\mathbf{q}_t\|_r \leq Ch^r \|u_t\|_{r+1}, \quad \text{and} \quad \|\boldsymbol{\theta}_{\sigma t}\| \leq Ch^r \|u_t\|_{r+1},$$

and hence, by Lemma 2.1,

$$\|\rho_{\mathbf{q}_t}\| \leq C(T)h^r (\|u_0\|_{r+2} + \|u_1\|_{r+1}).$$

Now, a use of the triangle inequality completes the proof of (3.7) for $j = 1$. For $j = 2$, differentiate again (3.5) with respect to t and repeat the above arguments to obtain the estimate.

For the second estimate (3.8), we use (3.1) for any $\mathbf{v} \in \mathbf{V}$ to arrive at

$$\begin{aligned} (\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{v}) &= (\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{v} - \Pi_h \mathbf{v}) + (\boldsymbol{\eta}_{\mathbf{q}}, \Pi_h \mathbf{v}) \\ &= (\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{v} - \Pi_h \mathbf{v}) - (\eta_u, \nabla \cdot \Pi_h \mathbf{v}) \\ &\leq \|\boldsymbol{\eta}_{\mathbf{q}}\| \|\mathbf{v} - \Pi_h \mathbf{v}\| + \|\eta_u\| \|\nabla \cdot \Pi_h \mathbf{v}\| \\ &\leq \left(Ch^2 (\|u_0\|_2 + \|u_1\|_1) + \|\eta_u\| \right) \|\mathbf{v}\|_{\mathbf{H}(\text{div}; \Omega)}, \end{aligned}$$

and hence, for nonzero $\mathbf{v} \in \mathbf{V}$

$$\frac{(\boldsymbol{\eta}_{\mathbf{q}}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}(\text{div}; \Omega)}} \leq Ch^2 (\|u_0\|_2 + \|u_1\|_1) + \|\eta_u\|.$$

By taking supremum over all nonzero $\mathbf{v} \in \mathbf{V}$, we obtain the desired estimate and this concludes the proof. \square

Now, we use a duality argument to estimate $\|\eta_u\|$.

Lemma 3.2 *Let $(\eta_u, \boldsymbol{\eta}_{\mathbf{q}}, \boldsymbol{\eta}_{\sigma})$ satisfy the system (3.1)-(3.3). Then, there is a positive constant C independent of h such that for $t \in (0, T]$*

$$(3.11) \quad \|D_t^j \eta_u(t)\| \leq Ch^2 (\|u_0\|_{2+j} + \|u_1\|_{1+j}), \quad j = 0, 1, 2,$$

and

$$(3.12) \quad \|\eta_u(t)\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| (\|u_0\|_3 + \|u_1\|_2).$$

Proof. Consider the auxiliary elliptic problem:

$$(3.13) \quad \nabla \cdot (A \nabla \zeta) = \eta_u \text{ in } \Omega,$$

$$(3.14) \quad \zeta = 0 \text{ on } \partial\Omega,$$

and set $\mathbf{p} = \nabla \zeta$ and $\boldsymbol{\psi} = A\mathbf{p}$ so that $\nabla \cdot \boldsymbol{\psi} = \eta_u$. Then, from the elliptic regularity result (2.2), it follows that

$$(3.15) \quad \|\zeta\|_2, \|\mathbf{p}\|_1, \|\boldsymbol{\psi}\|_1 \leq C \|\eta_u\|.$$

Clearly, the following system of equations is satisfied for all $(w, \mathbf{v}, \mathbf{z}) \in W \times \mathbf{V} \times \mathbf{V}$

$$(3.16) \quad (\mathbf{p}, \mathbf{v}) + (\zeta, \nabla \cdot \mathbf{v}) = 0,$$

$$(3.17) \quad (\boldsymbol{\psi}, \mathbf{z}) - (A\mathbf{p}, \mathbf{z}) = 0,$$

$$(3.18) \quad (\nabla \cdot \boldsymbol{\psi}, w) = (\eta_u, w).$$

Now, choose $w = \eta_u$, $\mathbf{z} = \boldsymbol{\eta}_{\mathbf{q}}$ and $\mathbf{v} = \boldsymbol{\eta}_{\boldsymbol{\sigma}}$ in (3.16)-(3.18), respectively, and then add the resulting equations to arrive at

$$(3.19) \quad \begin{aligned} \|\eta_u\|^2 &= (\nabla \cdot \boldsymbol{\psi}, \eta_u) + (\boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{q}}) - (A\mathbf{p}, \boldsymbol{\eta}_{\mathbf{q}}) + (\mathbf{p}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) + (\zeta, \nabla \cdot \boldsymbol{\eta}_{\boldsymbol{\sigma}}) \\ &= (\nabla \cdot (\boldsymbol{\psi} - \Pi_h \boldsymbol{\psi}), \eta_u) + (\boldsymbol{\psi} - \Pi_h \boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{q}}) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \boldsymbol{\eta}_{\mathbf{q}}) \\ &\quad + (\mathbf{p} - \Pi_h \mathbf{p}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) + (\zeta - P_h \zeta, \nabla \cdot \boldsymbol{\eta}_{\boldsymbol{\sigma}}) + (\nabla \cdot \Pi_h \boldsymbol{\psi}, \eta_u) \\ &\quad + (\Pi_h \boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{q}}) - (A(\Pi_h \mathbf{p}), \boldsymbol{\eta}_{\mathbf{q}}) + (\Pi_h \mathbf{p}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) + (P_h \zeta, \nabla \cdot \boldsymbol{\eta}_{\boldsymbol{\sigma}}). \end{aligned}$$

Next, set $\mathbf{v}_h = \Pi_h \boldsymbol{\psi}$, $\mathbf{z}_h = \Pi_h \mathbf{p}$ and $w_h = P_h \zeta$ in (3.1)-(3.3) with $\eta_u = \theta_u - \rho_u$. Then substitute in (3.19) and use the Cauchy-Schwarz inequality and (2.3) for $r = 1$ to obtain

$$\begin{aligned} \|\eta_u\|^2 &= (\nabla \cdot (\boldsymbol{\psi} - \Pi_h \boldsymbol{\psi}), \theta_u) + (\boldsymbol{\psi} - \Pi_h \boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{q}}) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \boldsymbol{\eta}_{\mathbf{q}}) \\ &\quad + (\mathbf{p} - \Pi_h \mathbf{p}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) + (\zeta - P_h \zeta, \nabla \cdot \boldsymbol{\eta}_{\boldsymbol{\sigma}}) - \int_0^t (B(t, s) \boldsymbol{\eta}_{\mathbf{q}}(s), \Pi_h \mathbf{p}) ds \\ &\leq C(h^2 \|u\|_2 \|\nabla \cdot \boldsymbol{\psi}\| + h \|\boldsymbol{\eta}_{\mathbf{q}}\| \|\nabla \cdot \boldsymbol{\psi}\| + h \|\boldsymbol{\eta}_{\mathbf{q}}\| \|\nabla \cdot \mathbf{p}\|) \\ &\quad + C(h^2 \|\nabla \cdot \mathbf{p}\| \|\nabla \cdot \boldsymbol{\sigma}\| + h^2 \|\zeta\|_2 \|\nabla \cdot \boldsymbol{\sigma}\|) + \int_0^t |(B(t, s) \boldsymbol{\eta}_{\mathbf{q}}(s), \Pi_h \mathbf{p})| ds. \end{aligned}$$

Note that for fixed s, t , we can use (2.3) for $r = 1$ to arrive at

$$(3.20) \quad \begin{aligned} (B(t, s) \boldsymbol{\eta}_{\mathbf{q}}(s), \Pi_h \mathbf{p}) &= -(B(t, s) \boldsymbol{\eta}_{\mathbf{q}}(s), \mathbf{p} - \Pi_h \mathbf{p}) + (\boldsymbol{\eta}_{\mathbf{q}}(s), B^*(t, s) \mathbf{p}) \\ &\leq C \|\boldsymbol{\eta}_{\mathbf{q}}(s)\| \|\mathbf{p} - \Pi_h \mathbf{p}\| + \|\boldsymbol{\eta}_{\mathbf{q}}(s)\|_{(\mathbf{H}(\text{div}; \Omega))^*} \|B^*(t, s) \mathbf{p}\|_{\mathbf{H}(\text{div}; \Omega)} \\ &\leq C(h \|\boldsymbol{\eta}_{\mathbf{q}}(s)\| + \|\boldsymbol{\eta}_{\mathbf{q}}(s)\|_{(\mathbf{H}(\text{div}; \Omega))^*}) \|\mathbf{p}\|_1, \end{aligned}$$

Hence,

$$\begin{aligned} \|\eta_u\|^2 &\leq C(h^2 \|u\|_2 + h \|\boldsymbol{\eta}_{\mathbf{q}}\| + h^2 \|\nabla \cdot \boldsymbol{\sigma}\|) (\|\nabla \cdot \boldsymbol{\psi}\| + \|\nabla \cdot \mathbf{p}\| + \|\zeta\|_2) \\ &\quad + C \left(\int_0^t (h \|\boldsymbol{\eta}_{\mathbf{q}}(s)\| + \|\boldsymbol{\eta}_{\mathbf{q}}(s)\|_{(\mathbf{H}(\text{div}; \Omega))^*}) ds \right) \|\mathbf{p}(t)\|_1. \end{aligned}$$

A use of estimates of $\|\boldsymbol{\eta}_{\mathbf{q}}\|$ from (3.7) with the elliptic regularity (3.15) yields

$$(3.21) \quad \|\eta_u\| \leq Ch^2(\|u_0\|_2 + \|u_1\|_1) + C \int_0^t \|\boldsymbol{\eta}_{\mathbf{q}}(s)\|_{(\mathbf{H}(\text{div}; \Omega))^*} ds.$$

Substituting (3.8) in (3.21), apply Gronwall's lemma to obtain

$$\|\eta_u\| \leq Ch^2(\|u_0\|_2 + \|u_1\|_1),$$

and this concludes the proof of (3.11) for $j = 0$.

In order to estimate $\|\eta_{u_t}\|$, consider again the elliptic problem (3.13)-(3.14) with replacing η_u on the right hand side of (3.13) by η_{u_t} . By setting

$$\mathbf{p} = \nabla \zeta, \quad \boldsymbol{\psi} = A\mathbf{p},$$

we have that

$$(\mathbf{p}, \mathbf{v}) + (\zeta, \nabla \cdot \mathbf{v}) = 0, \quad (\boldsymbol{\psi}, \mathbf{z}) - (A\mathbf{p}, \mathbf{z}) = 0, \quad \text{and} \quad (\nabla \cdot \boldsymbol{\psi}, w) = (\eta_{u_t}, w).$$

From the standard regularity results, it follows that

$$(3.22) \quad \|\zeta\|_2, \|\mathbf{p}\|_1, \|\psi\|_1 \leq C\|\eta_{u_t}\|.$$

Now, differentiate with respect to time the three equations in (3.1)-(3.1) to obtain

$$(3.23) \quad (\eta_{\mathbf{q}_t}, \mathbf{v}_h) + (\eta_{u_t}, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(3.24) \quad (\eta_{\sigma_t}, \mathbf{z}_h) - (A\eta_{\mathbf{q}_t}, \mathbf{z}_h) + (B(t, t)\eta_{\mathbf{q}}, \mathbf{z}_h) + \int_0^t (B_t(t, s)\eta_{\mathbf{q}}(s), \mathbf{z}_h) ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(3.25) \quad (\nabla \cdot \eta_{\sigma_t}, w_h) = 0, \quad w_h \in W_h.$$

Following the previous steps for deriving the estimate of $\|\eta_u\|$, a use of (3.22)-(3.25) at the appropriate steps leads to

$$(3.26) \quad \begin{aligned} \|\eta_{u_t}\|^2 &= (\nabla \cdot (\psi - \Pi_h \psi), \eta_{u_t}) + (\eta_{\mathbf{q}_t}, \psi - \Pi_h \psi) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \eta_{\mathbf{q}_t}) \\ &\quad + (\eta_{\sigma_t}, \mathbf{p} - \Pi_h \mathbf{p}) - (\eta_{\sigma_t}, \mathbf{p}) - (B(t, t)\eta_{\mathbf{q}}, \Pi_h \mathbf{p}) - \int_0^t (B_t(t, s)\eta_{\mathbf{q}}, \Pi_h \mathbf{p}) ds \\ &= (\nabla \cdot (\psi - \Pi_h \psi), \theta_{ut}) + (\eta_{\mathbf{q}_t}, \psi - \Pi_h \psi) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \eta_{\mathbf{q}_t}) \\ &\quad + (\eta_{\sigma_t}, \mathbf{p} - \Pi_h \mathbf{p}) + (B(t, t)\eta_{\mathbf{q}}, \mathbf{p} - \Pi_h \mathbf{p}) + \int_0^t (B_t(t, s)\eta_{\mathbf{q}}, \mathbf{p} - \Pi_h \mathbf{p}) ds \\ &\quad + (\zeta - P_h \zeta, \nabla \cdot \eta_{\sigma_t}) - (B(t, t)\eta_{\mathbf{q}}, \mathbf{p}) - \int_0^t (B_t(t, s)\eta_{\mathbf{q}}, \mathbf{p}) ds, \end{aligned}$$

where P_h is the L^2 -projection onto the conforming finite element space consisting of C^0 -piecewise linear elements which is a subspace of $H_0^1(\Omega)$. Since for all other terms except the last three terms on the right hand side of (3.26), it is easy to derive the estimates, it is enough to estimate the last three terms. An application of integration by parts shows that

$$|(\zeta - P_h \zeta, \nabla \cdot \eta_{\sigma_t})| = |(\nabla(\zeta - P_h \zeta), \eta_{\sigma_t})| \leq \|\nabla(\zeta - P_h \zeta)\| \|\eta_{\sigma_t}\| \leq Ch^2 \|u_t\|_2 \|\zeta\|_2.$$

For the last two terms on the right hand side of (3.26), we observe that

$$(B(t, t)\eta_{\mathbf{q}}, \mathbf{p}) \leq M \|\eta_{\mathbf{q}}\|_{(\mathbf{H}(\text{div}; \Omega))^*} \|\mathbf{p}\|_1,$$

and a use of integration by parts yields

$$\left| \int_0^t (B_t(t, s)\eta_{\mathbf{q}}, \mathbf{p}) ds \right| \leq M \left(\int_0^t \|\eta_{\mathbf{q}}(s)\|_{(\mathbf{H}(\text{div}; \Omega))^*} ds \right) \|\mathbf{p}(t)\|_1.$$

All together, we obtain using the Cauchy-Schwarz inequality, Lemmas 2.1, 3.1 and 3.2, the approximation properties (2.3) of projections P_h and Π_h , with elliptic regularity result (3.22) in (3.26) the following estimate

$$\|\eta_{u_t}\| \leq Ch^2 (\|u_0\|_3 + \|u_1\|_2).$$

Finally, by differentiating (3.23)-(3.25) with respect to time and following the previous steps we establish the estimate for $j = 2$.

In order to show the estimate (3.12), we write (3.1) as

$$(\eta_{\mathbf{q}}, \mathbf{v}_h) - (\rho_u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h.$$

Using Lemma 2.1 in [22], it follows that

$$\|\rho_u\|_{L^\infty(\Omega)} \leq C |\log h| \|\eta_{\mathbf{q}}\|,$$

for some constant C independent of h . A use of (2.4) and (3.7) completes the proof. \square

4 Semidiscrete Error Estimates

In this section, error estimates for the semidiscrete problem are derived. Using the mixed Ritz-Volterra projections defined in section 3, we rewrite

$$\begin{aligned} e_u &:= u - u_h = (u - \tilde{u}_h) - (u_h - \tilde{u}_h) =: \eta_u - \xi_u, \\ \mathbf{e}_q &:= \mathbf{q} - \mathbf{q}_h = (\mathbf{q} - \tilde{\mathbf{q}}_h) - (\mathbf{q}_h - \tilde{\mathbf{q}}_h) =: \boldsymbol{\eta}_q - \boldsymbol{\xi}_q, \\ \mathbf{e}_\sigma &:= \boldsymbol{\sigma} - \boldsymbol{\sigma}_h = (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) - (\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h) =: \boldsymbol{\eta}_\sigma - \boldsymbol{\xi}_\sigma, \end{aligned}$$

where, $(u, \mathbf{q}, \boldsymbol{\sigma})$ and $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h)$ are solutions of (1.13)-(1.15) and (2.6)-(2.8), respectively. Note that, $(e_u, \mathbf{e}_q, \mathbf{e}_\sigma)$ satisfy the following equations

$$(4.1) \quad (\mathbf{e}_q, \mathbf{v}_h) + (\mathbf{e}_u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(4.2) \quad (\mathbf{e}_\sigma, \mathbf{z}_h) - (A\mathbf{e}_q, \mathbf{z}_h) + \int_0^t (B(t, s)\mathbf{e}_q(s), \mathbf{z}_h) ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(4.3) \quad (\mathbf{e}_{u_{tt}}, w_h) - (\nabla \cdot \mathbf{e}_\sigma, w_h) = 0, \quad w_h \in W_h.$$

Since estimates of $\eta_u, \boldsymbol{\eta}_q$ and $\boldsymbol{\eta}_\sigma$ are known from Lemmas 3.1 and 3.2, it is sufficient to estimate $\xi_u, \boldsymbol{\xi}_q$ and $\boldsymbol{\xi}_\sigma$. Using (3.1)-(3.3), we rewrite (4.1)-(4.3) as

$$(4.4) \quad (\boldsymbol{\xi}_q, \mathbf{v}_h) + (\xi_u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(4.5) \quad (\boldsymbol{\xi}_\sigma, \mathbf{z}_h) - (A\boldsymbol{\xi}_q, \mathbf{z}_h) + \int_0^t (B(t, s)\boldsymbol{\xi}_q(s), \mathbf{z}_h) ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(4.6) \quad (\xi_{u_{tt}}, w_h) - (\nabla \cdot \boldsymbol{\xi}_\sigma, w_h) = (\eta_{u_{tt}}, w_h), \quad w_h \in W_h.$$

Below, one of the main results for the semidiscrete problem is proved.

Theorem 4.1 *Let $(u, \mathbf{q}, \boldsymbol{\sigma})$ and $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h)$ satisfy (1.13)-(1.15) and (2.6)-(2.8), respectively with $u_h(0) = P_h u_0$ and $u_{ht}(0) = P_h u_1$. Then, there exists a positive constant C independent of the discretizing parameter h such that for $t \in (0, T]$*

$$(4.7) \quad \|u_t(t) - u_{ht}(t)\| \leq Ch^2 (\|u_0\|_4 + \|u_1\|_3),$$

and

$$(4.8) \quad \|\mathbf{q}(t) - \mathbf{q}_h(t)\| + \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\| \leq Ch^2 (\|u_0\|_4 + \|u_1\|_3).$$

Proof. First differentiate (4.4) with respect to time and set $\mathbf{v}_h = \boldsymbol{\xi}_\sigma$ in the resulting equation, $\mathbf{z}_h = -\boldsymbol{\xi}_{q_t}$ in (4.5) and $w_h = \xi_{u_t}$ in (4.6). Then, add the resulting equations to arrive at

$$(4.9) \quad \frac{1}{2} \frac{d}{dt} \left(\|\xi_{u_t}\|^2 + \|A^{1/2} \boldsymbol{\xi}_q(t)\|^2 \right) = (\eta_{u_{tt}}, \xi_{u_t}) + \int_0^t (B(t, s) \boldsymbol{\xi}_q(s), \boldsymbol{\xi}_{q_t}) ds.$$

Apply integration by parts to the integral term on the right hand side of (4.9) to find that

$$(4.10) \quad \begin{aligned} \int_0^t B(t, s; \boldsymbol{\xi}_q(s), \boldsymbol{\xi}_{q_t}) ds &= \frac{d}{dt} \int_0^t (B(t, s) \boldsymbol{\xi}_q(s), \boldsymbol{\xi}_q) ds - (B(t, t) \boldsymbol{\xi}_q(t), \boldsymbol{\xi}_q) \\ &\quad - \int_0^t (B_t(t, s) \boldsymbol{\xi}_q(s), \boldsymbol{\xi}_q) ds. \end{aligned}$$

Substitute (4.10) in (4.9), then integrate the resulting equation from 0 to t . A use of the Cauchy-Schwarz inequality with the boundedness of B and the positive definite property of A yields

$$(4.11) \quad \begin{aligned} \|\xi_{u_t}(t)\|^2 + \|A^{1/2} \boldsymbol{\xi}_q(t)\|^2 &\leq \|\xi_{u_t}(0)\|^2 + \|A^{1/2} \boldsymbol{\xi}_q(0)\|^2 + 2 \int_0^t \|\eta_{u_{tt}}\| \|\xi_{u_t}\| ds \\ &\quad + C(T, a_0, a_1) \left(\int_0^t \|A^{1/2} \boldsymbol{\xi}_q(s)\| \|A^{1/2} \boldsymbol{\xi}_q(t)\| ds + \int_0^t \|A^{1/2} \boldsymbol{\xi}_q(s)\|^2 ds \right). \end{aligned}$$

For some $t^* \in [0, t]$, let

$$\left(\|\xi_{u_t}(t^*)\|^2 + \|A^{1/2}\xi_{\mathbf{q}}(t^*)\|^2 \right) = \max_{0 \leq s \leq t} \left(\|\xi_{u_t}(s)\|^2 + \|A^{1/2}\xi_{\mathbf{q}}(s)\|^2 \right).$$

Then, at $t = t^*$, (4.11) becomes

$$\begin{aligned} \|\xi_{u_t}(t^*)\| + \|A^{1/2}\xi_{\mathbf{q}}(t^*)\| &\leq \|\xi_{u_t}(0)\| + \|A^{1/2}\xi_{\mathbf{q}}(0)\| + 2 \int_0^{t^*} \|\eta_{u_{tt}}\| ds \\ &\quad + C(T, a_0, a_1) \int_0^{t^*} \|A^{1/2}\xi_{\mathbf{q}}(s)\| ds, \end{aligned}$$

and hence,

$$\begin{aligned} \|\xi_{u_t}(t)\| + \|A^{1/2}\xi_{\mathbf{q}}(t)\| &\leq \|\xi_{u_t}(t^*)\| + \|A^{1/2}\xi_{\mathbf{q}}(t^*)\| \\ &\leq \|\xi_{u_t}(0)\| + \|A^{1/2}\xi_{\mathbf{q}}(0)\| + 2 \int_0^t \|\eta_{u_{tt}}\| ds + C(T, a_0, a_1) \int_0^t \|A^{1/2}\xi_{\mathbf{q}}(s)\| ds. \end{aligned}$$

Now an application of Gronwall's lemma yields

$$(4.12) \quad \|\xi_{u_t}(t)\| + \|A^{1/2}\xi_{\mathbf{q}}(t)\| \leq C \left(\|\xi_{u_t}(0)\| + \|\xi_{\mathbf{q}}(0)\| + \int_0^t \|\eta_{u_{tt}}\| ds \right).$$

To estimate $\|\xi_{\sigma}\|$, choose $\mathbf{z}_h = \xi_{\sigma}$ in (4.5). Then, use the Cauchy-Schwarz inequality to arrive at

$$\|\xi_{\sigma}(t)\| \leq C \left(\|A^{1/2}\xi_{\mathbf{q}}\| + \int_0^t \|A^{1/2}\xi_{\mathbf{q}}(s)\| ds \right).$$

From the triangle inequality, we obtain

$$\|e_{u_t}\| \leq \|\xi_{u_t}\| + \|\eta_{u_t}\|.$$

Apply Lemmas 3.1 and 3.2 with the choices $u_h(0) = Pu_0$, $u_{ht}(0) = P_h u_1$ and $\mathbf{q}_h(0) = \Pi_h(\nabla u_0)$ to arrive at the estimate (4.7). In a similar way, we can establish (4.8) and this completes the rest of the proof. \square

As a consequence of Theorem 4.1, we have the following $L^\infty(L^\infty)$ estimate.

Corollary 4.1 *Assume that the mesh is quasi-uniform. Then, under the assumptions of Theorem 4.1, there exists a positive constant C independent of the discretizing parameter h such that for $t \in (0, T]$*

$$(4.13) \quad \|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| (\|u_0\|_4 + \|u_1\|_3).$$

Proof. Apply Lemma 2.1 in [22] to (4.4) to obtain

$$\|\xi_u(t)\|_{L^\infty(\Omega)} \leq C |\log h| \|\xi_{\mathbf{q}}\|.$$

Since

$$\|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq \|\eta_u(t)\|_{L^\infty(\Omega)} + \|\xi_u(t)\|_{L^\infty(\Omega)},$$

(4.13) follows from (3.12) and (4.12) and this completes the proof. \square

Remark 4.1 *As a consequence of Theorem 4.1 and the following inequality*

$$\|\xi_u(t)\| \leq C \left(\|\xi_u(0)\| + \int_0^t \|\xi_{u_t}\| ds \right),$$

we easily derive the following estimate of $u - u_h$:

$$(4.14) \quad \|u - u_h\|_{L^\infty(L^2(\Omega))} \leq Ch^2 (\|u_0\|_4 + \|u_1\|_3).$$

Since from (4.14), we obtain an optimal error estimate of $\|u - u_h\|_{L^\infty(L^2(\Omega))}$, when $u_0 \in H^4 \cap H_0^1$ and $u_1 \in H^3 \cap H_0^1$, we now use a variant of Baker's nonstandard formulation (see [2]) to provide a proof of $L^\infty(L^2)$ estimate of $u - u_h$ under reduced regularity conditions on u_0 and u_1 . More precisely, we shall obtain optimal $L^\infty(L^2)$ estimate for $u - u_h$, when $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. In the rest of this section, we make use of the following notation for $\bar{\phi}$:

$$\bar{\phi}(t) = \int_0^t \phi(s) ds.$$

After integrating (4.5) and (4.6) with respect to t , the following system of equations is derived

$$(4.15) \quad (\xi_{\mathbf{q}}, \mathbf{v}_h) + (\xi_u, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(4.16) \quad (\bar{\xi}_{\boldsymbol{\sigma}}, \mathbf{z}_h) - (A\bar{\xi}_{\mathbf{q}}, \mathbf{z}_h) + \int_0^t \left(\int_0^s (B(s, \tau) \xi_{\mathbf{q}}(\tau), \mathbf{z}_h) d\tau \right) ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(4.17) \quad (\xi_{u_t}, w_h) - (\nabla \cdot \bar{\xi}_{\boldsymbol{\sigma}}, w_h) = (\eta_{u_t}, w_h), \quad w_h \in W_h.$$

Note that in (4.17), we have used the fact that $u_{ht}(0) = P_h u_1$, that is, $(e_{ht}(0), w_h) = 0$ for all $w_h \in W_h$.

Theorem 4.2 *Let $(u, \mathbf{q}, \boldsymbol{\sigma})$ and $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h)$ satisfy (1.13)-(1.15) and (2.6)-(2.8), respectively, with $u_h(0) = P_h u_0$ and $u_{ht}(0) = P_h u_1$. Then, there exists a positive constant C independent of the discretizing parameter h such that for $t \in (0, T]$*

$$(4.18) \quad \|u(t) - u_h(t)\| \leq Ch^2 (\|u_0\|_3 + \|u_1\|_2).$$

Proof. Choose $\mathbf{v}_h = \bar{\xi}_{\boldsymbol{\sigma}}$, $\mathbf{z}_h = -\xi_{\mathbf{q}}$ and $w_h = \xi_u$, respectively in (4.15), (4.16) and (4.17). On adding the resulting equations, we arrive at

$$\frac{1}{2} \frac{d}{dt} [\|\xi_u\|^2 + \|A^{1/2} \bar{\xi}_{\mathbf{q}}\|^2] = (\eta_{u_t}, \xi_u) + \int_0^t \left(\int_0^s (B(s, \tau) \xi_{\mathbf{q}}(\tau), \xi_{\mathbf{q}}) d\tau \right) ds.$$

Integrate from 0 to t to deduce

$$(4.19) \quad \begin{aligned} \|\xi_u(t)\|^2 + \|A^{1/2} \bar{\xi}_{\mathbf{q}}(t)\|^2 &= \|\xi_u(0)\|^2 + 2 \int_0^t (\eta_{u_t}, \xi_u) ds \\ &\quad - 2 \int_0^t \int_0^s \int_0^\tau (B(\tau, \tau^*) \xi_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau ds. \end{aligned}$$

Let I denote the last term on the right hand side of (4.19). Integration by parts yields

$$\begin{aligned} I &= -2 \int_0^t \int_0^s (B(\tau, \tau) \bar{\xi}_{\mathbf{q}}(\tau), \xi_{\mathbf{q}}(s)) d\tau ds + 2 \int_0^t \int_0^s \int_0^\tau (B_{\tau^*}(\tau, \tau^*) \bar{\xi}_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau ds \\ &= -2I_1 + 2I_2. \end{aligned}$$

For I_1 , we again integrate by parts in time so that

$$\begin{aligned} |I_1| &= \left| \int_0^t (B(s, s) \bar{\xi}_{\mathbf{q}}(s), \bar{\xi}_{\mathbf{q}}(t)) ds - \int_0^t (B(s, s) \bar{\xi}_{\mathbf{q}}(s), \bar{\xi}_{\mathbf{q}}(s)) ds \right| \\ &\leq \frac{a_1}{a_0} \left\{ \|A^{1/2} \bar{\xi}_{\mathbf{q}}(t)\| \int_0^t \|A^{1/2} \bar{\xi}_{\mathbf{q}}(s)\| ds + \int_0^t \|A^{1/2} \bar{\xi}_{\mathbf{q}}(s)\|^2 ds \right\}. \end{aligned}$$

Similarly for I_2 , we note that

$$\begin{aligned} |I_2| &= \left| \int_0^t \int_0^s (B_\tau(s, \tau) \bar{\xi}_{\mathbf{q}}(\tau), \bar{\xi}_{\mathbf{q}}(t)) d\tau ds - \int_0^t \int_0^s (B_\tau(s, \tau) \bar{\xi}_{\mathbf{q}}(\tau), \bar{\xi}_{\mathbf{q}}(s)) d\tau ds \right| \\ &\leq \frac{a_1 T}{a_0} \left\{ \|A^{1/2} \bar{\xi}_{\mathbf{q}}(t)\| \int_0^t \|A^{1/2} \bar{\xi}_{\mathbf{q}}(s)\| ds + \int_0^t \|A^{1/2} \bar{\xi}_{\mathbf{q}}(s)\|^2 ds \right\}. \end{aligned}$$

Using the Cauchy-Schwarz inequality and the bounds for I_1 and I_2 , we obtain

$$\begin{aligned} \|\xi_u(t)\|^2 &+ \|A^{1/2}\bar{\xi}_{\mathbf{q}}(t)\|^2 \leq \|\xi_u(0)\|^2 + 2 \int_0^t \|\eta_{u_t}(s)\| \|\xi_u(s)\| ds \\ &+ C(T, a_0, a_1) \left(\|A^{1/2}\bar{\xi}_{\mathbf{q}}(t)\| \int_0^t \|A^{1/2}\bar{\xi}_{\mathbf{q}}(s)\| ds + \int_0^t \|A^{1/2}\bar{\xi}_{\mathbf{q}}(s)\|^2 ds \right). \end{aligned}$$

Now, let $|||(\xi_u, \bar{\xi}_{\mathbf{q}})(t)|||^2 = \|\xi_u(t)\|^2 + \|A^{1/2}\bar{\xi}_{\mathbf{q}}(t)\|^2$ and

$$|||(\xi_u, \bar{\xi}_{\mathbf{q}})(t^*)||| = \max_{0 \leq s \leq t} |||(\xi_u, \bar{\xi}_{\mathbf{q}})(s)|||,$$

for some $t^* \in [0, t]$. Then, at $t = t^*$, we note that

$$\begin{aligned} |||(\xi_u, \bar{\xi}_{\mathbf{q}})(t^*)||| &\leq |||(\xi_u, \bar{\xi}_{\mathbf{q}})(0)||| + 2 \int_0^{t^*} \|\eta_{u_t}(s)\| ds \\ &+ C(T, a_0, a_1) \int_0^{t^*} |||(\xi_u, \bar{\xi}_{\mathbf{q}})(s)||| ds, \end{aligned}$$

and therefore,

$$\begin{aligned} |||(\xi_u, \bar{\xi}_{\mathbf{q}})(t)||| &\leq |||(\xi_u, \bar{\xi}_{\mathbf{q}})(0)||| + 2 \int_0^t \|\eta_{u_t}(s)\| ds \\ &+ C(T, a_0, a_1) \int_0^t |||(\xi_u, \bar{\xi}_{\mathbf{q}})(s)||| ds. \end{aligned}$$

An application of Gronwall's lemma yields

$$\|\xi_u(t)\| + \|A^{1/2}\bar{\xi}_{\mathbf{q}}(t)\| \leq C \left(\|\xi_u(0)\| + \int_0^t \|\eta_{u_t}\| ds \right).$$

Finally, a use of the triangle inequality with Lemma 3.2 concludes the proof of Theorem 4.2. \square

5 Error Estimates for a Completely Discrete Scheme

In this section, we introduce further notations and formulate a completely discrete scheme by applying an implicit finite difference method to discretize the time variable of the semidiscrete system (2.6)-(2.8). Then, we discuss optimal error estimates.

Let k ($0 < k < 1$) be the time step, $k = T/N$ for some positive integer N , and $t_n = nk$. For any function ϕ of time, let ϕ^n denote $\phi(t_n)$. We shall use this notation for functions defined for continuous in time as well as those defined for discrete in time. Set

$$\phi^{n+1/2} = \frac{\phi^{n+1} + \phi^n}{2}, \quad \phi^{n;1/4} = \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4} = \frac{\phi^{n+1/2} + \phi^{n-1/2}}{2},$$

and define the following terms for the difference quotients:

$$\begin{aligned} \partial_t \phi^{n+1/2} &= \frac{\phi^{n+1} - \phi^n}{k}, \quad \bar{\partial}_t \phi^{n+1/2} = \frac{\phi^{n+1/2} - \phi^{n-1/2}}{k}, \\ \delta_t \phi^n &= \frac{\phi^{n+1} - \phi^{n-1}}{2k} = \frac{\partial_t \phi^{n+1/2} + \partial_t \phi^{n-1/2}}{2}, \end{aligned}$$

and

$$\partial_t^2 \phi^n = \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{2k} = \frac{\partial_t \phi^{n+1/2} - \partial_t \phi^{n-1/2}}{k}.$$

The discrete-in-time scheme is based on a symmetric difference approximation around the nodal points, and integral terms are computed by using the second order quadrature formula

$$\epsilon^n(\phi) = k \sum_{j=0}^{n-1} g(t_{j+1/2}) \approx \int_0^{t_n} g(s) ds, \quad \text{with } t_{j+1/2} = (j + 1/2)k.$$

The quadrature error $\mathcal{E}^n(g)$ is defined by

$$\mathcal{E}^n(g) = \epsilon^n(g) - \int_0^{t_n} g(s) ds = \sum_{j=0}^{n-1} \left(kg^{j+1/2} - \int_{t_j}^{t_{j+1}} g(s) ds \right).$$

Using Peano's kernel theorem, see [16], it can be written as

$$\mathcal{E}^n(g) = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \psi(s) D_s^2 g(s) ds$$

with

$$\psi(s) = \begin{cases} (s - t_j)(s - t_{j+1/2}), & s \in [t_j, t_{j+1/2}], \\ (s - t_{j+1})(s - t_{j+1/2}), & s \in [t_{j+1/2}, t_{j+1}]. \end{cases}$$

Now, let $\mathcal{B}(t, s; \cdot, \cdot) : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$ be the bilinear form defined by

$$\mathcal{B}(t, s; \phi, \chi) = (B(t, s)\phi(s), \chi).$$

Then, the discrete-in-time scheme for the problem (1.13)-(1.15) is to seek $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in W_h \times \mathbf{V}_h \times \mathbf{V}_h$, such that

$$(5.1) \quad \frac{2}{k}(\partial_t U^{1/2}, w_h) - (\nabla \cdot \mathbf{Z}^{1/2}, w_h) = \left(\frac{2}{k}u_1, w_h\right), \quad w_h \in W_h,$$

$$(5.2) \quad (\mathbf{Q}^{n+1/2}, \mathbf{v}_h) + (U^{n+1/2}, \nabla \cdot \mathbf{v}_h) = 0, \quad n \geq 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(5.3) \quad (\mathbf{Z}^{n+1/2}, \mathbf{z}_h) - (A\mathbf{Q}^{n+1/2}, \mathbf{z}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \mathbf{z}_h)) = 0, \quad n \geq 0, \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(5.4) \quad (\partial_t^2 U^n, w_h) - (\nabla \cdot \mathbf{Z}^{n;1/4}, w_h) = 0, \quad n \geq 1, \quad w_h \in W_h,$$

with given initial data $(U^0, \mathbf{Q}^0, \mathbf{Z}^0)$ in $W_h \times \mathbf{V}_h \times \mathbf{V}_h$. Here, in (5.3),

$$\epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \mathbf{z}_h)) = \frac{1}{2}(\epsilon^{n+1}(\mathcal{B}^{n+1}(\mathbf{Q}, \mathbf{z}_h)) + \epsilon^n(\mathcal{B}^n(\mathbf{Q}, \mathbf{z}_h))),$$

where

$$\epsilon^n(\mathcal{B}^n(\mathbf{Q}, \chi)) = k \sum_{j=0}^{n-1} (B(t_n, t_{j+1/2})\mathbf{Q}^{j+1/2}, \chi).$$

This choice of the time discretization leads to a second order accuracy in k .

For $\phi \in \mathbf{V}_h$, we define a linear functional $\mathcal{E}_B^n(\phi)$ representing the error in the quadrature formula by

$$\mathcal{E}_B^n(\phi)(\chi) = \epsilon^n(\mathcal{B}^n(\phi, \chi)) - \int_0^{t_n} \mathcal{B}(t_n, s; \phi, \chi) ds.$$

Notice that $\mathcal{E}_B^0(\phi) = 0$. In our analysis, we shall use the following lemma, which can be found in [16].

Lemma 5.1 *There exists a positive constant C independent of h and k such that the following estimates hold:*

$$k \sum_{n=0}^m \|\mathcal{E}_B^{n+1}(\phi)\| \leq Ck^2 \int_0^{t_{m+1}} (\|\phi\| + \|\phi_t\| + \|\phi_{tt}\|) ds,$$

and

$$k \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\phi)\| \leq Ck^2 \int_0^{t_{m+1}} (\|\phi\| + \|\phi_t\| + \|\phi_{tt}\|) ds.$$

Define $e_U^n := u^n - U^n$, $\mathbf{e}_Q^n := \mathbf{q}^n - \mathbf{Q}^n$ and $\mathbf{e}_Z^n := \boldsymbol{\sigma}^n - \mathbf{Z}^n$. From (5.1)-(5.4) and (1.13)-(1.15), we derive the system of equations

$$(5.5) \quad \frac{2}{k}(\partial_t e_U^{1/2}, w_h) - (\nabla \cdot \mathbf{e}_Z^{1/2}, w_h) = -(2r^0, w_h), \quad w_h \in W_h,$$

$$(5.6) \quad (\mathbf{e}_Q^{n+1/2}, \mathbf{v}_h) + (e_U^{n+1/2}, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(5.7) \quad (\mathbf{e}_Z^{n+1/2}, \mathbf{z}_h) - (A\mathbf{e}_Q^{n+1/2}, \mathbf{z}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{e}_Q, \mathbf{z}_h)) = \mathcal{E}_B^{n+1/2}(\mathbf{q})(\mathbf{z}_h), \quad \mathbf{z}_h \in \mathbf{V}_h,$$

$$(5.8) \quad (\partial_t^2 e_U^n, w_h) - (\nabla \cdot \mathbf{e}_Z^{n;1/4}, w_h) = -(r^n, w_h), \quad w_h \in W_h,$$

where $r^0 = \frac{1}{2}u_{tt}^{1/2} + \frac{1}{k}(u_t(0) - \partial_t u^{1/2})$ and

$$(5.9) \quad r^n = u_{tt}^{n;1/4} - \partial_t^2 u^n = \frac{1}{12} \int_{-k}^k (|t| - k) (3 - 2(1 - |t|/k)^2) \frac{\partial^4 u}{\partial t^4}(t^n + t) dt, \quad n \geq 1.$$

In order to derive *a priori* error estimates for the completely discrete scheme, we rewrite

$$\begin{aligned} e_U^n &= (u^n - \tilde{u}_h^n) - (U^n - \tilde{u}_h^n) =: \eta_U^n - \xi_U^n, \\ \mathbf{e}_Q^n &= (\mathbf{q}^n - \tilde{\mathbf{q}}_h^n) - (\mathbf{Q}^n - \tilde{\mathbf{q}}_h^n) =: \boldsymbol{\eta}_Q^n - \boldsymbol{\xi}_Q^n, \\ \mathbf{e}_Z^n &= (\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n) - (\mathbf{Z}^n - \tilde{\boldsymbol{\sigma}}_h^n) =: \boldsymbol{\eta}_Z^n - \boldsymbol{\xi}_Z^n. \end{aligned}$$

Since estimates for η_U , $\boldsymbol{\eta}_Q$ and $\boldsymbol{\eta}_Z$ are known from Lemmas 3.1 and 3.2, it is sufficient to estimate ξ_U , ξ_Q and ξ_Z . From (5.5)-(5.8), we obtain the following system

$$(5.10) \quad \frac{2}{k}(\partial_t \xi_U^{1/2}, w_h) - (\nabla \cdot \boldsymbol{\xi}_Z^{1/2}, w_h) = \frac{2}{k}(\partial_t \eta_U^{1/2}, w_h) + (2r^0, w_h),$$

$$(5.11) \quad (\boldsymbol{\xi}_Q^{n+1/2}, \mathbf{v}_h) + (\xi_U^{n+1/2}, \nabla \cdot \mathbf{v}_h) = 0,$$

$$(5.12) \quad (\boldsymbol{\xi}_Z^{n+1/2}, \mathbf{z}_h) - (A\boldsymbol{\xi}_Q^{n+1/2}, \mathbf{z}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\boldsymbol{\xi}_Q, \mathbf{z}_h)) = -\mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)(\mathbf{z}_h),$$

$$(5.13) \quad (\partial_t^2 \xi_U^n, w_h) - (\nabla \cdot \boldsymbol{\xi}_Z^{n;1/4}, w_h) = (\partial_t^2 \eta_U^n, w_h) + (r^n, w_h).$$

Below, one of the main theorems of this section is proved.

Theorem 5.1 *Let Ω be a bounded convex polygonal domain in \mathbb{R}^2 and let $(u, \mathbf{q}, \boldsymbol{\sigma})$ be the solution of (1.13)-(1.15). Further, let $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in W_h \times \mathbf{V}_h \times \mathbf{V}_h$ be the solution of (5.1)-(5.4). Assume that $U^0 = \tilde{u}_h(0)$ and $\mathbf{Q}^0 = P_h \nabla u_0$. Then there exists constants $C > 0$, independent of h and k , and $k_0 > 0$, such that for $0 < k < k_0$ and $m = 0, 1, \dots, N-1$*

$$(5.14) \quad \|\partial_t(u(t_{m+1/2}) - U^{m+1/2})\| \leq C(h^2 + k^2)(\|u_0\|_4 + \|u_1\|_3),$$

and

$$(5.15) \quad \|\mathbf{q}(t_{m+1/2}) - \mathbf{Q}^{m+1/2}\| + \|\boldsymbol{\sigma}(t_{m+1/2}) - \mathbf{Z}^{m+1/2}\| \leq C(h^2 + k^2)(\|u_0\|_4 + \|u_1\|_3).$$

Proof. Write (5.11) at $n-1/2$ and subtract the resulting one from (5.11). Take averages of (5.12) at two time levels, that is, at $n+1/2$ and $n-1/2$. Then choose $\mathbf{v}_h = \boldsymbol{\xi}_Z^{n;1/4}$ in the modified equation (5.11), $\mathbf{z}_h = -\delta_t \boldsymbol{\xi}_Q^n$ in the modified equation (5.12) and $w_h = \delta_t \xi_U^n$ in (5.13). Then add to obtain

$$\begin{aligned} (5.16) \quad & \frac{1}{2} \bar{\partial}_t \left(\|\partial_t \xi_U^{n+1/2}\|^2 + \|A^{1/2} \boldsymbol{\xi}_Q^{n+1/2}\|^2 \right) = (\partial_t^2 \eta_U^n + r^n, \delta_t \xi_U^n) + \mathcal{E}_B^{n;1/4}(\tilde{\mathbf{q}}_h)(\delta_t \boldsymbol{\xi}_Q^n) \\ & + \frac{1}{2} \left(\epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\boldsymbol{\xi}_Q, \delta_t \boldsymbol{\xi}_Q^n)) + \epsilon^{n-1/2}(\mathcal{B}^{n-1/2}(\boldsymbol{\xi}_Q, \delta_t \boldsymbol{\xi}_Q^n)) \right) \\ & = I_1^n + I_2^n + I_3^n. \end{aligned}$$

With $\Psi^n = (\xi_U^n, \boldsymbol{\xi}_Q^n)$, define

$$|||\Psi^{n+1/2}|||^2 = \|\partial_t \xi_U^{n+1/2}\|^2 + \|A^{1/2} \boldsymbol{\xi}_Q^{n+1/2}\|^2.$$

Multiply (5.16) by $2k$ and sum from $n = 2$ to m to obtain

$$(5.17) \quad |||\Psi^{m+1/2}|||^2 \leq |||\Psi^{3/2}|||^2 + 2k \left| \sum_{n=2}^m (I_1^n + I_2^n + I_3^n) \right|.$$

Define for some m^* with $0 \leq m^* \leq m$,

$$|||\Psi^{m^*+1/2}||| = \max_{0 \leq n \leq m} |||\Psi^{n+1/2}|||.$$

A use of the Cauchy-Schwarz inequality yields

$$\begin{aligned} 2k \left| \sum_{n=2}^m I_1^n \right| &\leq \sum_{n=2}^m (\|\partial_t^2 \eta_u^n\| + \|r^n\|) \left(\|\partial_t \xi_U^{n+1/2}\| + \|\partial_t \xi_U^{n-1/2}\| \right) \\ &\leq 2k \sum_{n=2}^m (\|\partial_t^2 \eta_u^n\| + \|r^n\|) |||\Psi^{m^*+1/2}|||. \end{aligned}$$

Setting

$$\tilde{B}_{j+1/2}^{n+1/2} = \frac{1}{2} (B(t_{n+1}, t_{j+1/2}) + B(t_n, t_{j+1/2})),$$

we now estimate one of the term of I_3^n . Note that

$$\begin{aligned} &\epsilon^{n+1/2} (\mathcal{B}^{n+1/2}(\xi_Q, \delta_t \xi_Q^n)) \\ &= \frac{k}{2} \left[\sum_{j=0}^n \mathcal{B}(t_{n+1}, t_{j+1/2}; \xi_Q^{j+1/2}, \delta_t \xi_Q^n) + \sum_{j=0}^{n-1} \mathcal{B}(t_n, t_{j+1/2}; \xi_Q^{j+1/2}, \delta_t \xi_Q^n) \right] \\ &= \frac{1}{2} \left(B(t_{n+1}, t_{n+1/2}) \xi_Q^{n+1/2}, (\xi_Q^{n+1/2} - \xi_Q^{n-1/2}) \right) + \sum_{j=0}^{n-1} \left(\tilde{B}_{j+1/2}^{n+1/2} \xi_Q^{j+1/2}, (\xi_Q^{n+1/2} - \xi_Q^{n-1/2}) \right) \\ &= I_{31}^n + I_{32}^n. \end{aligned}$$

For the second term I_{32}^n , we use the fact that

$$(5.18) \quad H^{n+1/2} \bar{\partial}_t \xi_Q^{n+1/2} = \bar{\partial}_t (H^{n+1/2} \xi_Q^{n+1/2}) - \bar{\partial}_t (H^{n+1/2}) \xi_Q^{n-1/2},$$

to obtain

$$\begin{aligned} I_{32}^n &= k \bar{\partial}_t \left(\sum_{j=0}^{n-1} \left(\tilde{B}_{j+1/2}^{n+1/2} \xi_Q^{n+1/2}, \xi_Q^{j+1/2} \right) \right) + \left(\tilde{B}_{n-1/2}^{n-1/2} \xi_Q^{n-1/2}, \xi_Q^{n-1/2} \right) \\ &\quad - k \sum_{j=0}^{n-1} \left(\bar{\partial}_t \left(\tilde{B}_{j+1/2}^{n+1/2} \right) \xi_Q^{j+1/2}, \xi_Q^{n-1/2} \right), \end{aligned}$$

and hence after summing up from $n = 2$ to m and multiplying by k

$$\begin{aligned} k \left| \sum_{n=2}^m I_{32}^n \right| &\leq k^2 \left| \sum_{j=0}^{m-1} \left(\tilde{B}_{j+1/2}^{m+1/2} \xi_Q^{j+1/2}, \xi_Q^{m+1/2} \right) - \left(\tilde{B}_{1/2}^{3/2} \xi_Q^{3/2}, \xi_Q^{1/2} \right) \right| + k \left| \sum_{n=2}^m \left(\tilde{B}_{n-1/2}^{n-1/2} \xi_Q^{n-1/2}, \xi_Q^{n-1/2} \right) \right| \\ &\quad + k^2 \left| \sum_{n=2}^m \sum_{j=0}^{n-1} \left(\bar{\partial}_t \left(\tilde{B}_{j+1/2}^{n+1/2} \right) \xi_Q^{j+1/2}, \xi_Q^{n-1/2} \right) \right| \\ &\leq a_1 k^2 \|\xi_Q^{m+1/2}\| \sum_{j=0}^{m-1} \|\xi_Q^{j+1/2}\| + a_1 k^2 \|\xi_Q^{3/2}\| \|\xi_Q^{1/2}\| + a_1 (1+T) k \sum_{j=0}^{m-1} \|\xi_Q^{j+1/2}\|^2 \\ &\leq C(a_0, a_1, T) k \left(\sum_{n=0}^{m-1} \|A^{1/2} \xi_Q^{n+1/2}\| \right) |||\Psi^{m^*+1/2}|||. \end{aligned}$$

All together, we obtain

$$k \left| \sum_{n=2}^m \epsilon^{n+1/2} (\mathcal{B}^{n+1/2}(\xi_Q, \delta_t \xi_Q^n)) \right| \leq \frac{a_1}{a_0^2} k \|\xi_Q^{m+1/2}\|^2 + C(a_0, a_1, T) k \left(\sum_{n=0}^{m-1} \|A^{1/2} \xi_Q^{n+1/2}\| \right) |||\Psi^{m^*+1/2}|||.$$

We can now estimate $\epsilon^{n-1/2}(\mathcal{B}^{n-1/2}(\xi_Q, \delta_t \xi_Q^n))$ in a similar way, but without having the term $\|A^{1/2} \xi_Q^{m+1/2}\|^2$ on the right hand side and thus, we arrive at

$$k \left| \sum_{n=2}^m I_3^n \right| \leq \frac{a_1 k}{a_0^2} \|A^{1/2} Q^{m+1/2}\|^2 + C(a_1, a_0, T) k \left(\sum_{n=0}^{m-1} \|A^{1/2} Q^{n+1/2}\| \right) |||\Psi^{m^*+1/2}|||.$$

In a similar way, using (5.18), the term I_2^n can be estimated as follows

$$k \left| \sum_{n=2}^m I_2^n \right| \leq \left| \mathcal{E}_B^{n;1/4}(\tilde{\mathbf{q}}_h)(\xi_Q^{m+1/2}) - \mathcal{E}_B^{1;1/4}(\tilde{\mathbf{q}}_h)(\xi_Q^{3/2}) - k \sum_{n=2}^m \bar{\partial}_t(\mathcal{E}_B^{n;1/4}(\tilde{\mathbf{q}}_h))(\xi_Q^{n-1/2}) \right|.$$

Notice that, since $\mathcal{E}_B^0 = 0$, it follows that

$$\mathcal{E}_B^n(\tilde{\mathbf{q}}_h)(\xi_Q^n) = k \sum_{j=0}^{n-1} \partial_t \mathcal{E}_B^{j+1/2}(\tilde{\mathbf{q}}_h)(\xi_Q^n),$$

and hence, we obtain

$$k \left| \sum_{n=2}^m I_2^n \right| \leq C(a_0, a_1) k \left(\sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \right) |||\Psi^{m^*+1/2}|||.$$

It remains now to bound the term $|||\Psi^{3/2}|||$ on the right hand side of (5.17). Take (5.16) at $n = 1$ to obtain

$$\begin{aligned} |||\Psi^{3/2}|||^2 &\leq |||\Psi^{1/2}|||^2 + 2k \left| (\partial_t^2 \eta_U^1 + r^1, \delta_t \xi_U^1) + \frac{1}{2} \left(\epsilon^{3/2} (\mathcal{B}^{3/2}(\xi_Q, \delta_t \xi_Q^1)) + \epsilon^{1/2} (\mathcal{B}^{1/2}(\xi_Q, \delta_t \xi_Q^1)) \right) \right. \\ &\quad \left. + \mathcal{E}_B^{1;1/4}(\tilde{\mathbf{q}}_h)(\delta_t \xi_Q^1) \right| \\ (5.19) \quad &\leq |||\Psi^{1/2}|||^2 + C(a_0, a_1) k \left(\|\partial_t^2 \eta_U^1\| + \|r^1\| + \|A^{1/2} \xi_Q^{1/2}\| \right. \\ &\quad \left. + \|A^{1/2} \xi_Q^{3/2}\| + \|\partial_t \mathcal{E}_B^{1/2}(\tilde{\mathbf{q}}_h)\| + \|\partial_t \mathcal{E}_B^{3/2}(\tilde{\mathbf{q}}_h)\| \right) |||\Psi^{m^*+1/2}|||. \end{aligned}$$

Substitution of estimates involving I_1^n, \dots, I_3^n and (5.19) in (5.17) yields

$$\begin{aligned} \left(1 - \frac{a_1}{a_0^2} k\right) |||\Psi^{m+1/2}|||^2 &\leq |||\Psi^{1/2}|||^2 + Ck \left\{ \sum_{n=1}^m \left(\|\partial_t^2 \eta_U^n\| + \|r^n\| \right) \right. \\ (5.20) \quad &\quad \left. + \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| + \sum_{n=0}^{m-1} |||\Psi^{n+1/2}||| \right\} |||\Psi^{m^*+1/2}|||, \end{aligned}$$

Choose $k_0 > 0$ such that for $0 < k \leq k_0$, $(1 - \frac{a_1}{a_0^2} k) > 0$. Then replace m by m^* in (5.20) and obtain after cancellation of $|||\Psi^{m^*+1/2}|||$ from the both sides

$$\begin{aligned} |||\Psi^{m+1/2}||| &\leq |||\Psi^{m^*+1/2}||| \leq C \left\{ |||\Psi^{1/2}||| + k \sum_{n=1}^m \left(\|\partial_t^2 \eta_U^n\| + \|r^n\| \right) \right. \\ &\quad \left. + \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| + \sum_{n=0}^{m-1} |||\Psi^{n+1/2}||| \right\}. \end{aligned}$$

Then an application of the discrete Gronwall's lemma yields

$$(5.21) \quad |||\Psi^{m+1/2}||| \leq C \left\{ |||\Psi^{1/2}||| + k \sum_{n=1}^m (||\partial_t^2 \eta_U^n|| + ||r^n||) + k \sum_{n=0}^m ||\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)|| \right\}.$$

To estimate $\Psi^{1/2}$ on the right hand side of this inequality, we first note that, $\xi_U^0 = 0$ since $U^0 = \tilde{u}_h(0)$, and hence, $\xi_U^{1/2} = \frac{k}{2} \partial_t \xi_U^{1/2}$. Now, choose $w_h = \xi_U^{1/2}$ in (5.10), $w_h = \xi_Z^{1/2}$ in (5.11), and $w_h = -\xi_Q^{1/2}$ in (5.12). Adding the resulting equations, and taking into account that $\epsilon^0(\phi) = 0$ and $\mathcal{E}_B^0(\phi) = 0$, we arrive at

$$\begin{aligned} |||\Psi^{1/2}|||^2 &\leq \left| \left(\partial_t \eta_U^{1/2}, \partial_t \xi_U^{1/2} \right) \right| + k \left| \left(r^0, \partial_t \xi_U^{1/2} \right) \right| + \frac{1}{2} \left| \epsilon^1(\mathcal{B}^1(\xi_Q, \xi_Q^{1/2})) \right| + \frac{1}{2} \left| \mathcal{E}_B^1(\tilde{\mathbf{q}}_h)(\xi_Q^{1/2}) \right| \\ &\leq C(a_0) \left(||\partial_t \eta_U^{1/2}|| + k ||r^0|| + ||\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)|| \right) |||\Psi^{1/2}||| + \frac{a_1 k}{2a_0} \|A^{1/2} \xi_Q^{1/2}\|^2 \\ &\leq C(a_0) \left(||\partial_t \eta_U^{1/2}|| + k ||r^0|| + ||\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)|| \right) |||\Psi^{1/2}||| + \frac{a_1 k}{2a_0} |||\Psi^{1/2}|||^2 \end{aligned}$$

For $0 < k \leq k_0$, $(1 - (a_1 k)/(2a_0)) > 0$ and hence, we obtain

$$|||\Psi^{1/2}||| \leq C \left\{ ||\partial_t \eta_U^{1/2}|| + k ||r^0|| + ||\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)|| \right\}.$$

Thus,

$$(5.22) \quad |||\Psi^{m+1/2}||| \leq C \left\{ ||\partial_t \eta_U^{1/2}|| + k \sum_{n=1}^m ||\partial_t^2 \eta_U^n|| + k \sum_{n=0}^m ||r^n|| + k \sum_{n=0}^m ||\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)|| \right\}.$$

To estimate the first two terms on the right hand side of (5.22), it is observed that

$$(5.23) \quad ||\partial_t \eta_U^{1/2}|| \leq \frac{1}{k} \int_0^k ||\eta_{Ut}(s)|| ds,$$

and a use of Taylor series expansions yields

$$\begin{aligned} k \sum_{n=1}^m ||\partial_t^2 \eta_U^n|| &\leq \frac{1}{k} \sum_{n=1}^m \left\{ \int_{t_n}^{t_{n+1}} (t_{n+1} - s) ||\eta_{Utt}(s)|| ds + \int_{t_{n-1}}^{t_n} (s - t_{n-1}) ||\eta_{Utt}(s)|| ds \right\} \\ (5.24) \quad &\leq 2 \int_0^{t_{m+1}} ||\eta_{Utt}(s)|| ds \leq C(T) h^2 (||u_0||_4 + ||u_1||_3). \end{aligned}$$

Further, from (5.9) it follows that

$$||r^n|| \leq Ck \int_{t_{n-1}}^{t_{n+1}} ||D_t^4 u(s)|| ds, \quad n \geq 1,$$

and

$$||r^0|| \leq Ck ||u_{ttt}||_{L^\infty(0, k/2; L^2(\Omega))} \leq Ck \int_0^{t_{m+1}} (||D_t^3 u(s)|| + ||D_t^4 u(s)||) ds.$$

Hence,

$$(5.25) \quad k \sum_{n=0}^m ||r^n|| \leq Ck^2 \int_0^{t_{m+1}} (||D_t^3 u(s)|| + ||D_t^4 u(s)||) ds \leq C(T) k^2 (||u_0||_4 + ||u_1||_3).$$

For the last term in (5.22), a use of Lemma 5.1 with the triangle inequality yields

$$k \sum_{n=0}^m ||\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)|| \leq Ck^2 \sum_{j=0}^2 \int_0^{t_{m+1}} (||D_t^j \mathbf{q}(s)|| + ||D_t^j \eta_{\mathbf{q}}(s)||) ds,$$

and hence, the estimates in Lemmas 2.1 and 3.1 show that

$$(5.26) \quad k \sum_{n=0}^m \|\partial_t \mathcal{E}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)\| \leq C(T)k^2(\|u_0\|_3 + \|u_1\|_2) + C(T)k^2h(\|u_0\|_4 + \|u_1\|_3).$$

Substitute (5.23)-(5.26) in (5.22) and use the triangle inequality with the results in Lemmas 3.1 and 3.2, to obtain the error estimates involving u and \mathbf{q} in (5.14)-(5.15). Now to complete the remaining estimate in (5.15), we need to estimate $\|\xi_{\mathcal{Z}}^{m+1/2}\|$. To do so, choose $\mathbf{z}_h = \xi_{\mathcal{Z}}^{m+1/2}$ in (5.12), and conclude that

$$\|\xi_{\mathcal{Z}}^{m+1/2}\| \leq C(a_1, T) \left(\max_{0 \leq n \leq m} \|\xi_{\mathcal{Q}}^{n+1/2}\| + \|\mathcal{E}_{\mathcal{B}}^{m+1/2}(\tilde{\mathbf{q}}_h)\| \right).$$

Finally, a use of the triangle inequality with Lemmas 3.1 and 3.2 completes the rest of the proof. \square

Below, we again recall a variant of Baker's nonstandard energy formulation to prove $\ell^\infty(L^2)$ -estimate for the error $\|e_U^n\|$ under reduced regularity conditions on the initial data.

Now introduce the following notations for proving the next theorem. Define

$$\hat{\phi}^0 = 0, \quad \hat{\phi}^n = k \sum_{j=0}^{n-1} \phi^{j+1/2}.$$

Then,

$$\partial_t \hat{\phi}^{n+1/2} = \phi^{n+1/2},$$

and

$$k \sum_{j=1}^n \phi^{j;1/4} = \hat{\phi}^{n+1/2} - \frac{k}{2} \phi^{1/2}.$$

Notice that

$$(5.27) \quad \begin{aligned} k \sum_{n=0}^m \psi^{n+1} \phi^{n+1/2} &= k \sum_{n=0}^m \psi^{n+1} \partial_t \hat{\phi}^{n+1/2} \\ &= \sum_{n=0}^m (\psi^{n+1} \hat{\phi}^{n+1} - \psi^n \hat{\phi}^n) - \sum_{n=0}^m (\psi^{n+1} - \psi^n) \hat{\phi}^n \\ &= \psi^{m+1} \hat{\phi}^{m+1} - k \sum_{n=0}^m \partial_t \psi^{n+1/2} \hat{\phi}^n. \end{aligned}$$

and similarly,

$$(5.28) \quad k \sum_{n=0}^m \psi^{n+1/2} \phi^{n+1/2} = \psi^{m+1/2} \hat{\phi}^{m+1} - k \sum_{n=0}^m \bar{\partial}_t \psi^{n+1/2} \hat{\phi}^n.$$

Multiplying (5.4) by k , summing over n , and taking into account (5.1), we obtain the new equation

$$(5.29) \quad (\partial_t U^{n+1/2}, w_h) - (\nabla \cdot \hat{\mathbf{Z}}^{n+1/2}, w_h) = (u_1, w_h).$$

The key idea here, which differs from Baker's approach, is that we compare the above equation with

$$(5.30) \quad (u_t, w) - (\nabla \cdot \bar{\boldsymbol{\sigma}}, w) = (u_1, w)$$

which is derived by integrating (1.15) with respect to time, where $\bar{\boldsymbol{\sigma}}(t) = \int_0^t \boldsymbol{\sigma}(s) ds$. By taking the average of (5.30) at t^{n+1} and t^n and using (5.29) we arrive at

$$(\partial_t e_U^{n+1/2}, w) - (\nabla \cdot \hat{\boldsymbol{\sigma}}^{n+1/2} - \nabla \cdot \hat{\mathbf{Z}}^{n+1/2}, w) = -(r_1^n, w) - (\nabla \cdot \hat{\boldsymbol{\sigma}}^{n+1/2} - \nabla \cdot \bar{\boldsymbol{\sigma}}^{n+1/2}, w),$$

with $r_1^n = u_t^{n+1/2} - \partial_t u^{n+1/2}$. Thus,

$$(\partial_t e_U^{n+1/2}, w) - (\nabla \cdot \hat{\mathbf{e}}_{\mathbf{Z}}^{n+1/2}, w) = -(r_1^n, w) - \mathcal{E}_{\mathcal{I}}^{n+1/2}(\nabla \cdot \boldsymbol{\sigma})(w_h),$$

where

$$\mathcal{E}_{\mathcal{I}}^n(\phi)(\chi) = (\hat{\phi}^n - \bar{\phi}^n, \chi) = k \sum_{j=0}^{n-1} (\phi^{j+1/2}, \chi) - \int_0^{t_n} (\phi(s), \chi) ds.$$

Using the definitions of \tilde{u}_h and $\tilde{\boldsymbol{\sigma}}_h$, it follows that

$$(\partial_t \xi_U^{n+1/2}, w_h) - (\nabla \cdot \hat{\boldsymbol{\xi}}_{\mathbf{Z}}^{n+1/2}, w_h) = (\partial_t \eta_U^{n+1/2}, w_h) + (r_1^n, w_h) + \mathcal{E}_{\mathcal{I}}^{n+1/2}(\nabla \cdot \boldsymbol{\sigma})(w_h).$$

To complete the system of error equations, we multiply (5.12) by k , sum over n , and take the average of the resulting equations. Including (5.12), we end up with following system

$$(5.31) \quad (\xi_Q^{n+1/2}, \mathbf{v}_h) + (\xi_U^{n+1/2}, \nabla \cdot \mathbf{v}_h) = 0,$$

$$(5.32) \quad (\hat{\boldsymbol{\xi}}_{\mathbf{Z}}^{n+1/2}, \mathbf{z}_h) - (A \hat{\boldsymbol{\xi}}_{\mathbf{Q}}^{n+1/2}, \mathbf{z}_h) + \hat{\epsilon}^{n+1/2}(\xi_Q, \mathbf{z}_h) = -\hat{\mathcal{E}}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)(\mathbf{z}_h),$$

$$(5.33) \quad (\partial_t \xi_U^{n+1/2}, w_h) - (\nabla \cdot \hat{\boldsymbol{\xi}}_{\mathbf{Z}}^{n+1/2}, w_h) = (\partial_t \eta_U^{n+1/2}, w_h) + (r_1^n, w_h) + \mathcal{E}_{\mathcal{I}}^{n+1/2}(\nabla \cdot \boldsymbol{\sigma})(w_h),$$

where

$$\hat{\epsilon}^{n+1/2}(\xi_Q, \mathbf{z}_h) = \frac{1}{2} \left[k \sum_{j=0}^n \epsilon^{j+1/2}(\mathcal{B}^{j+1/2}(\xi_Q, \mathbf{z}_h)) + k \sum_{j=0}^{n-1} \epsilon^{j+1/2}(\mathcal{B}^{j+1/2}(\xi_Q, \mathbf{z}_h)) \right],$$

and

$$\hat{\mathcal{E}}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)(\mathbf{z}_h) = \frac{1}{2} \left[k \sum_{j=0}^n \mathcal{E}_{\mathcal{B}}^{j+1/2}(\tilde{\mathbf{q}}_h)(\mathbf{z}_h) + k \sum_{j=0}^{n-1} \mathcal{E}_{\mathcal{B}}^{j+1/2}(\tilde{\mathbf{q}}_h)(\mathbf{z}_h) \right].$$

Theorem 5.2 *Let Ω be a bounded convex polygonal domain in \mathbb{R}^2 , and let $(u, \mathbf{q}, \boldsymbol{\sigma})$ be the solution of (1.13)-(1.15). Further, let $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in W_h \times \mathbf{V}_h \times \mathbf{V}_h$ be the solution of (5.1)-(5.4). With $U^0 = P_h u_0$ and $\mathbf{Q}^0 = P_h \nabla u_0$, there exists a positive constant C , independent of h and k , such that for small k with $k = O(h)$, the following estimate holds:*

$$(5.34) \quad \|u(t_{m+1}) - U^{m+1}\| \leq C(h^2 + k^2) (\|u_0\|_3 + \|u_1\|_2), \quad m = 0, 1, \dots, N-1.$$

Proof. Choose $w_h = \hat{\boldsymbol{\xi}}_{\mathbf{Z}}^{n+1/2}$ in (5.31), $\mathbf{z}_h = -\boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}$ in (5.32), and $v_h = \xi_U^{n+1/2}$ in (5.33). Adding the resulting equations, we find that

$$\begin{aligned} (\partial_t \xi_U^{n+1/2}, \xi_U^{n+1/2}) &+ (A \hat{\boldsymbol{\xi}}_{\mathbf{Q}}^{n+1/2}, \boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}) = (\partial_t \eta_U^{n+1/2}, \xi_U^{n+1/2}) + (r_1^n, \xi_U^{n+1/2}) \\ &- \hat{\epsilon}^{n+1/2}(\xi_Q, \boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}) - \hat{\mathcal{E}}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)(\boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}) + \mathcal{E}_{\mathcal{I}}^{n+1/2}(\nabla \cdot \boldsymbol{\sigma})(\xi_U^{n+1/2}). \end{aligned}$$

Multiply both sides by $2k$ and sum from $n = 0$ to m . Then, use that $\hat{\boldsymbol{\xi}}_{\mathbf{Q}}^0 = 0$, $\boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2} = \partial_t \hat{\boldsymbol{\xi}}_{\mathbf{Q}}^{n+1/2}$, and

$$(\partial_t \xi_U^{n+1/2}, \xi_U^{n+1/2}) = \frac{1}{2k} (\|\xi_U^{n+1}\|^2 - \|\xi_U^n\|^2),$$

to arrive at

$$\begin{aligned} \|\xi_U^{m+1}\|^2 &+ \|A^{1/2} \hat{\boldsymbol{\xi}}_{\mathbf{Q}}^{m+1}\|^2 = \|\xi_U^0\|^2 - 2k \sum_{n=0}^m \hat{\epsilon}^{n+1/2}(\xi_Q, \boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}) - 2k \sum_{n=0}^m \hat{\mathcal{E}}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)(\boldsymbol{\xi}_{\mathbf{Q}}^{n+1/2}) \\ &+ 2k \sum_{n=0}^m (\partial_t \eta_U^{n+1/2}, \xi_U^{n+1/2}) + 2k \sum_{n=0}^m (r_1^n, \xi_U^{n+1/2}) + 2k \sum_{n=0}^m \mathcal{E}_{\mathcal{I}}^{n+1/2}(\nabla \cdot \boldsymbol{\sigma})(\xi_U^{n+1/2}) \\ (5.35) \quad &= \|\xi_U^0\|^2 + I_1^m + I_2^m + I_3^m + I_4^m + I_5^m. \end{aligned}$$

For convenience, we use the following notations: $B_r^s = B(t_s, t_r)$, $\bar{\partial}_i \phi_{i+1/2} = (\phi_{i+1/2} - \phi_{i-1/2})/k$, and denote by $\psi \cdot \phi$ the L^2 inner product of ψ and ϕ . Further, let $|||(\xi_U^n, \hat{\xi}_Q^n)|||^2 = \|\xi_U^n\|^2 + \|A^{1/2} \hat{\xi}_Q^n\|^2$ and for some $m^* \in [0; m+1]$, define

$$|||(\xi_U^{m^*}, \hat{\xi}_Q^{m^*})||| = \max_{0 \leq n \leq m+1} |||(\xi_U^n, \hat{\xi}_Q^n)|||.$$

To estimate the first term I_1^m , we observe that by (5.28)

$$\begin{aligned} \epsilon^{j+1}(B^{j+1}(\xi_Q, \xi_Q^{n+1/2})) &= k \sum_{i=0}^j B_{i+1/2}^{j+1} \xi_Q^{i+1/2} \cdot \xi_Q^{n+1/2} \\ &= B_{j+1/2}^{j+1} \hat{\xi}_Q^{j+1} \cdot \xi_Q^{n+1/2} - k \sum_{i=0}^j \left(\bar{\partial}_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i \cdot \xi_Q^{n+1/2}. \end{aligned}$$

Setting

$$\Theta^{n+1} = \sum_{j=0}^n B_{j+1/2}^{j+1} \hat{\xi}_Q^{j+1} \quad \text{and} \quad \Upsilon^{n+1} = k \sum_{j=0}^n \sum_{i=0}^j \left(\bar{\partial}_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i,$$

we now obtain

$$(5.36) \quad k^2 \sum_{n=0}^m \sum_{j=0}^n \epsilon^{j+1}(B^{j+1}(\xi_Q, \xi_Q^{n+1/2})) = k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \xi_Q^{n+1/2} - k^2 \sum_{n=0}^m \Upsilon^{n+1} \cdot \xi_Q^{n+1/2}.$$

Next, we estimate the terms Θ^{n+1} and Υ^{n+1} . Using (5.27) we have

$$k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \xi_Q^{n+1/2} = k \Theta^{m+1} \cdot \hat{\xi}_Q^{m+1} - k^2 \sum_{n=0}^m \partial_t \Theta^{n+1/2} \cdot \hat{\xi}_Q^n,$$

and hence, using the definition of Θ^n , it follows that

$$(5.37) \quad k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \xi_Q^{n+1/2} = k \sum_{n=0}^m B_{n+1/2}^{n+1} \hat{\xi}_Q^{n+1} \cdot \hat{\xi}_Q^{m+1} - k \sum_{n=0}^m B_{n+1/2}^{n+1} \hat{\xi}_Q^{n+1} \cdot \hat{\xi}_Q^n.$$

Similarly,

$$\begin{aligned} k^2 \sum_{n=0}^m \Upsilon^{n+1} \cdot \xi_Q^{n+1/2} &= k \Upsilon^{m+1} \cdot \hat{\xi}_Q^{m+1} - k^2 \sum_{n=0}^m \partial_t \Upsilon^{n+1/2} \cdot \hat{\xi}_Q^n \\ &= k^2 \left(\sum_{j=0}^m \sum_{i=0}^j \left(\bar{\partial}_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i \right) \cdot \hat{\xi}_Q^{m+1} \\ &\quad - k^2 \sum_{n=0}^m \left(\sum_{i=0}^n \left(\bar{\partial}_i(B_{i+1/2}^{n+1}) \right) \hat{\xi}_Q^i \right) \cdot \hat{\xi}_Q^n. \end{aligned}$$

On substitution of (5.37) and (5.38) in (5.36), a use of Cauchy-Schwarz inequality with $\|D_{t,s} B(t, s)\| \leq a_1$, yields

$$\left| k^2 \sum_{n=0}^m \sum_{j=0}^n \epsilon^{j+1}(B^{j+1}(\xi_Q, \xi_Q^{n+1/2})) \right| \leq \frac{a_1 k}{a_0} \|A^{1/2} \hat{\xi}_Q^{m+1}\|^2 + C(a_0, a_1, T) \left(k \sum_{n=0}^m \|A^{1/2} \hat{\xi}_Q^n\| \right) |||(\xi_U^{m^*}, \hat{\xi}_Q^{m^*})|||.$$

Since, similar bounds can be obtained for other terms in $k \sum_{n=0}^m \hat{\epsilon}^{n+1/2}(\xi_Q, \xi_Q^{n+1/2})$, we finally conclude that

$$|I_1^m| \leq \frac{a_1 k}{2a_0} \|A^{1/2} \hat{\xi}_Q^{m+1}\| + C(a_0, a_1, T) \left(k \sum_{n=0}^m \|A^{1/2} \hat{\xi}_Q^n\| \right) |||(\xi_U^{m^*}, \hat{\xi}_Q^{m^*})|||.$$

Now, with $\Lambda^{n+1} = \sum_{j=0}^n \mathcal{E}_{\mathcal{B}}^{j+1}(\tilde{\mathbf{q}}_h)$, we observe that

$$\begin{aligned} k^2 \sum_{n=0}^m \sum_{j=0}^n \mathcal{E}_{\mathcal{B}}^{j+1}(\tilde{\mathbf{q}}_h)(\xi_{\mathcal{Q}}^{n+1/2}) &= k^2 \sum_{n=0}^m \Lambda^{n+1} \partial_t \hat{\xi}_{\mathcal{Q}}^{n+1/2} \\ &= k \Lambda^{m+1}(\hat{\xi}_{\mathcal{Q}}^{m+1}) - k \sum_{n=0}^m (\Lambda^{n+1} - \Lambda^n)(\hat{\xi}_{\mathcal{Q}}^n) \\ &= k \sum_{j=0}^m \mathcal{E}_{\mathcal{B}}^{j+1}(\tilde{\mathbf{q}}_h)(\hat{\xi}_{\mathcal{Q}}^{m+1}) - k \sum_{n=0}^m \mathcal{E}_{\mathcal{B}}^{n+1}(\tilde{\mathbf{q}}_h)(\hat{\xi}_{\mathcal{Q}}^n). \end{aligned}$$

Since, the terms in $k \sum_{n=0}^m \hat{\mathcal{E}}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)(\xi_{\mathcal{Q}}^{n+1/2})$ have a similar form, we deduce that

$$|I_2^m| \leq C(a_0)k \sum_{n=0}^m (||\mathcal{E}_{\mathcal{B}}^{n+1}(\tilde{\mathbf{q}}_h)||) |||(\xi_U^{m*}, \hat{\xi}_{\mathcal{Q}}^{m*})|||.$$

For I_3^m and I_4^m , we have

$$|I_3^m + I_4^m| \leq C(a_0)k \sum_{n=0}^m \left(\left| \partial_t \eta_U^{n+1/2} \right| + \|r_1^n\| \right) |||(\xi_U^{m*}, \hat{\xi}_{\mathcal{Q}}^{m*})|||.$$

To estimate the last term I_5^m , we first notice that $(\nabla \cdot \sigma, \xi_U^{n+1/2}) = (\nabla \cdot \tilde{\sigma}_h, \xi_U^{n+1/2})$ since $\xi_U^{n+1/2} \in W_h$. Then by (5.31)

$$(\nabla \cdot \sigma, \xi_U^{n+1/2}) = -(\xi_{\mathcal{Q}}^{n+1/2}, \tilde{\sigma}_h).$$

Thus, it follows that

$$\begin{aligned} -k \sum_{n=0}^m \mathcal{E}_{\mathcal{I}}^{n+1}(\nabla \cdot \sigma)(\xi_U^{n+1/2}) &= k \sum_{n=0}^m \mathcal{E}_{\mathcal{I}}^{n+1}(\tilde{\sigma}_h)(\xi_{\mathcal{Q}}^{n+1/2}) \\ &= \mathcal{E}_{\mathcal{I}}^{m+1}(\tilde{\sigma}_h)(\hat{\xi}_{\mathcal{Q}}^{m+1}) - k \sum_{n=0}^m \partial_t \mathcal{E}_{\mathcal{I}}^{n+1/2}(\tilde{\sigma}_h)(\hat{\xi}_{\mathcal{Q}}^n). \end{aligned}$$

Similarly, we have

$$-k \sum_{n=0}^m \mathcal{E}_{\mathcal{I}}^n(\nabla \cdot \sigma)(\xi_U^{n+1/2}) = \mathcal{E}_{\mathcal{I}}^m(\tilde{\sigma}_h)(\hat{\xi}_{\mathcal{Q}}^{m+1}) - k \sum_{n=1}^m \partial_t \mathcal{E}_{\mathcal{I}}^{n-1/2}(\tilde{\sigma}_h)(\hat{\xi}_{\mathcal{Q}}^n),$$

which yields

$$|I_5^m| \leq C(a_0) \left(\|\mathcal{E}_{\mathcal{I}}^{m+1/2}(\tilde{\sigma}_h)\| + k \sum_{n=0}^m \|\partial_t \mathcal{E}_{\mathcal{I}}^{n+1/2}(\tilde{\sigma}_h)\| \right) |||(\xi_U^{m*}, \hat{\xi}_{\mathcal{Q}}^{m*})|||.$$

On substituting the above estimates in (5.35) and following steps in previous theorems, we arrive at

$$\begin{aligned} \left(1 - (a_1/2a_0)k\right) |||(\xi_U^{m+1}, \hat{\xi}_{\mathcal{Q}}^{m+1})||| &\leq Ck \sum_{n=0}^m \left(\left| \partial_t \eta_U^{n+1/2} \right| + \|r_1^n\| + \|\mathcal{E}_{\mathcal{B}}^{n+1}(\tilde{\mathbf{q}}_h)\| + \|\partial_t \mathcal{E}_{\mathcal{I}}^{n+1/2}(\tilde{\sigma}_h)\| \right. \\ &\quad \left. + \|\partial_t \mathcal{E}_{\mathcal{I}}^{n+1/2}(\tilde{\sigma}_h)\| \right) + C\|\mathcal{E}_{\mathcal{I}}^{m+1/2}(\tilde{\sigma}_h)\| + \|\xi_U^0\| \\ &\leq Ck \sum_{n=0}^m \left(\left| \partial_t \eta_U^{n+1/2} \right| + \|r_1^n\| + \|\mathcal{E}_{\mathcal{B}}^{n+1}(\tilde{\mathbf{q}}_h)\| + \|\partial_t \mathcal{E}_{\mathcal{I}}^{n+1/2}(\tilde{\sigma}_h)\| \right. \\ &\quad \left. + |||(\xi_U^n, \hat{\xi}_{\mathcal{Q}}^n)||| \right) + C\|\mathcal{E}_{\mathcal{I}}^{m+1/2}(\tilde{\sigma}_h)\| + \|\xi_U^0\|. \end{aligned}$$

Since for $0 < k \leq k_0$, $(1 - (a_1/2a_0)k)$ can be made positive, an application of the discrete Gronwall's lemma yields

$$(5.38) \quad \|\xi_U^{m+1}\| + \|A^{1/2}\hat{\xi}_Q^{m+1}\| \leq Ck \sum_{n=0}^m \left(\left\| \partial_t \eta_U^{n+1/2} \right\| + \|r_1^n\| + \|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)\| + \|\partial_t \mathcal{E}_I^{n+1/2}(\tilde{\sigma}_h)\| \right) + C \left(\|\mathcal{E}_I^{m+1/2}(\tilde{\sigma}_h)\| + \|\xi_U^0\| \right).$$

The first two terms on the right hand side can be bounded as follows:

$$k \sum_{n=0}^m \left\| \partial_t \eta_U^{n+1/2} \right\| \leq \int_0^{t_{m+1}} \|\eta_{Ut}(s)\| ds,$$

and

$$k \sum_{n=0}^m \|r_1^n\| \leq Ck^2 \int_0^{t_{m+1}} \|D_t^3 u(s)\| ds.$$

For the third term, we note that $\tilde{\mathbf{q}}_h = -(\mathbf{q} - \tilde{\mathbf{q}}_h) + \mathbf{q} = -\boldsymbol{\eta}_Q + \mathbf{q}$, and hence

$$(5.39) \quad k \sum_{n=0}^m \|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)\| \leq k \sum_{n=0}^m \|\mathcal{E}_B^{n+1}(\boldsymbol{\eta}_Q)\| + k \sum_{n=0}^m \|\mathcal{E}_B^{n+1}(\mathbf{q})\|.$$

For the last term on the right hand side of (5.39), use Lemmas 5.1 and 2.1 to obtain

$$(5.40) \quad \begin{aligned} k \sum_{n=0}^m \|\mathcal{E}_B^{n+1}(\mathbf{q})\| &\leq Ck^2 \int_0^{t_{m+1}} (\|\mathbf{q}\| + \|\mathbf{q}_t\| + \|\mathbf{q}_{tt}\|) ds \\ &\leq Ck^2 (\|u_0\|_3 + \|u_1\|_2). \end{aligned}$$

For the first term on the right hand side of (5.39), we note that

$$(5.41) \quad \begin{aligned} \mathcal{E}_B^{n+1}(\boldsymbol{\eta}_Q)(\chi) &= \epsilon^{n+1} (\mathcal{B}^{n+1}(\boldsymbol{\eta}_Q, \chi)) - \int_0^{t_{n+1}} \mathcal{B}(t_{n+1}, s; \boldsymbol{\eta}_Q, \chi) ds \\ &= \sum_{j=0}^n \left[k \left(B(t_{n+1}, t_{j+1/2}) \boldsymbol{\eta}_Q^{j+1/2}, \chi \right) - \int_{t_j}^{t_{j+1}} \mathcal{B}(t_{n+1}, s; \boldsymbol{\eta}_Q, \chi) ds \right] \\ &= \sum_{j=0}^n \left(k B(t_{n+1}, t_{j+1/2}) \boldsymbol{\eta}_Q^{j+1/2} - \int_{t_j}^{t_{j+1}} B(t_{n+1}, s) \boldsymbol{\eta}_Q(s) ds, \chi \right). \end{aligned}$$

From the midpoint quadrature error, it follows that

$$kg^{j+1/2} - \int_{t_j}^{t_{j+1}} g(s) ds = \int_{t_j}^{t_{j+1/2}} (s - t_j)(s - t_{j+1/2}) D_s^2 g(s) ds + \int_{t_{j+1/2}}^{t_{j+1}} (s - t_{j+1})(s - t_{j+1/2}) D_s^2 g(s) ds.$$

Use integration by parts to find that the boundary terms become zero and therefore, we arrive at

$$kg^{j+1/2} - \int_{t_j}^{t_{j+1}} g(s) ds = - \int_{t_j}^{t_{j+1/2}} (2s - (t_{j+1/2} + t_j)) D_s g(s) ds - \int_{t_{j+1/2}}^{t_{j+1}} (2s - (t_{j+1} + t_{j+1/2})) D_s g(s) ds$$

and

$$\left| kg^{j+1/2} - \int_{t_j}^{t_{j+1}} g(s) ds \right| \leq \frac{k}{2} \int_{t_j}^{t_{j+1}} |D_s g(s)| ds,$$

Thus, (5.41) with (3.7) leads to

$$|\mathcal{E}_B^{n+1}(\boldsymbol{\eta}_Q)(\chi)| \leq Ck \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \left(\|\boldsymbol{\eta}_Q(s)\| + \|\boldsymbol{\eta}_{Qt}(s)\| \right) ds \|\chi\|,$$

and

$$\begin{aligned}
\|\mathcal{E}_B^{n+1}(\boldsymbol{\eta}_Q)\| &\leq Ck \int_0^{t_{n+1}} (\|\boldsymbol{\eta}_Q(s)\| + \|\boldsymbol{\eta}_{Q_t}(s)\|) ds \\
(5.42) \qquad \qquad \qquad &\leq Ckht_{n+1}(\|u_0\|_3 + \|u_1\|_2).
\end{aligned}$$

On substitution of (5.40) and (5.42) in (5.39), it follows that

$$k \sum_{n=0}^m \|\mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \leq C(T)(kh + k^2)(\|u_0\|_3 + \|u_1\|_2).$$

Following similar line of proof, we can easily show that the last two terms in (5.38) are also bounded by $C(T)(kh + k^2)(\|u_0\|_3 + \|u_1\|_2)$. Finally, by using the triangle inequality and the estimates in Lemmas 3.1 and 3.2 we complete the proof of the theorem. \square

6 Error Estimates for the Standard Mixed Method

Now, we extend our analysis to discuss optimal error estimates for u and $\boldsymbol{\sigma}$, satisfying the standard mixed method (1.10)-(1.11) with minimal regularity of the initial data. We first recall the following regularity results.

Lemma 6.1 *Let $(u, \boldsymbol{\sigma})$ satisfy (1.10)-(1.11). Then,*

$$\|D_t^j u(t)\| + \|D_t^{j-1} u(t)\|_1 + \|D_t^{j-1} \boldsymbol{\sigma}(t)\| \leq C(T)(\|u_0\|_j + \|u_1\|_{j-1}), \quad j = 1, \dots, 4,$$

and

$$\|D_t^j u(t)\|_2 \leq C(T)(\|u_0\|_{j+2} + \|u_1\|_{j+1}), \quad j = 0, 1, 2.$$

With W_h and \mathbf{V}_h defined as in Section 2, we define the corresponding semidiscrete mixed finite element approximation to (1.10)-(1.11) as: Find a pair $(u_h, \boldsymbol{\sigma}_h) \in W_h \times \mathbf{V}_h$ such that

$$(6.1) \quad (\alpha \boldsymbol{\sigma}_h, \mathbf{v}_h) + \int_0^t (M(t, s) \boldsymbol{\sigma}_h(s), \mathbf{v}_h) ds + (\nabla \cdot \mathbf{v}_h, u_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$

$$(6.2) \quad (u_{htt}, w_h) - (\nabla \cdot \boldsymbol{\sigma}_h, w_h) = 0 \quad \forall w_h \in W_h,$$

with $u_h(0) = P_h u_0$, and $u_{ht}(0) = P_h u_1$.

Below, we present the main theorem of this section.

Theorem 6.1 *Let $(u, \boldsymbol{\sigma})$ and $(u_h, \boldsymbol{\sigma}_h)$ satisfy (1.10)-(1.11) and (6.1)-(6.2), respectively, with $u_h(0) = P_h u_0$ and $u_{ht}(0) = P_h u_1$. Then, there exists a positive constant C , independent of h , such that for $t \in (0, T]$*

$$\|u(t) - u_h(t)\| \leq Ch^2 (\|u_0\|_3 + \|u_1\|_2).$$

6.1 Mixed Ritz-Volterra projections

We discuss some of the properties of the mixed Ritz-Volterra projections used in our analysis. We formulate the mixed Ritz-Volterra projections as follows. Given $(u(t), \boldsymbol{\sigma}(t)) \in W \times \mathbf{V}$, for $t \in (0, T]$, find $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h) : (0, T] \rightarrow W_h \times \mathbf{V}_h$ satisfying

$$(6.3) \quad (\alpha \boldsymbol{\eta}_\sigma, \mathbf{v}_h) + \int_0^t (M(t, s) \boldsymbol{\eta}_\sigma(s), \mathbf{v}_h) ds + (\nabla \cdot \mathbf{v}_h, \eta_u) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(6.4) \quad (\nabla \cdot \boldsymbol{\eta}_\sigma, w_h) = 0, \quad w_h \in W_h,$$

where $\eta_u := (u - \tilde{u}_h)$ and $\boldsymbol{\eta}_\sigma := (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)$.

The following lemma can be easily obtained by combining Theorem 2.6 of [11] and Lemma 6.1.

Lemma 6.2 *Let $(\eta_u, \boldsymbol{\eta}_\sigma)$ satisfy the system (6.3)-(6.4). Then, there is a positive constant C independent of h such that*

$$(6.5) \quad \|D_t^j \boldsymbol{\eta}_\sigma(t)\| \leq Ch^r (\|u_0\|_{j+r+1} + \|u_1\|_{j+r}), \quad j = 0, 1, 2, \quad r = 1, 2,$$

and

$$(6.6) \quad \|D_t^j \eta_u(t)\| \leq Ch^2 (\|u_0\|_{j+2} + \|u_1\|_{j+1}), \quad j = 0, 1, 2.$$

6.2 Error Estimates

Here, we discuss the proof of Theorem 6.1. Set

$$e_u := u - u_h = \eta_u - \xi_u \quad \text{and} \quad \mathbf{e}_\sigma := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\eta}_\sigma - \boldsymbol{\xi}_\sigma,$$

with $\xi_u = u_h - \tilde{u}_h$ and $\boldsymbol{\xi}_\sigma = \boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h$. Then, (e_u, \mathbf{e}_σ) satisfy the following error equations

$$(6.7) \quad (\alpha \mathbf{e}_\sigma, \mathbf{v}_h) + \int_0^t (M(t, s) \mathbf{e}_\sigma(s), \mathbf{v}_h) ds + (\nabla \cdot \mathbf{v}_h, e_u) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$$(6.8) \quad (e_{utt}, w_h) - (\nabla \cdot \mathbf{e}_\sigma, w_h) = 0 \quad \forall w_h \in W_h.$$

Using the mixed Ritz-Volterra projections, we write the above equations in terms of $\xi_u, \boldsymbol{\xi}_\sigma$ as

$$(6.9) \quad (\alpha \boldsymbol{\xi}_\sigma, \mathbf{v}_h) + \int_0^t (M(t, s) \boldsymbol{\xi}_\sigma(s), \mathbf{v}_h) ds + (\nabla \cdot \mathbf{v}_h, \xi_u) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$$(6.10) \quad (\xi_{utt}, w_h) - (\nabla \cdot \boldsymbol{\xi}_\sigma, w_h) = (\eta_{utt}, w_h) \quad \forall w_h \in W_h.$$

Below, we present a proof of our main theorem.

Proof of Theorem 6.1:

Since the estimate η_u is known from Lemma 6.2, it is enough to estimate ξ_u . We first integrate (6.10) and use the fact that $u_{ht}(0) = P_h u_1$ to obtain

$$(6.11) \quad (\xi_{u_t}, w_h) - (\nabla \cdot \bar{\boldsymbol{\xi}}_\sigma, w_h) = (\eta_{u_t}, w_h) \quad \forall w_h \in W_h.$$

Now, choose $v_h = \bar{\boldsymbol{\xi}}_\sigma$ and $w_h = \xi_u$ in (6.9) and (6.11), respectively, then add the resulting equations and use integration by parts to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|\xi_u\|^2 + \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma\|^2] &= (\eta_{u_t}, \xi_u) - \int_0^t (M(t, s) \boldsymbol{\xi}_\sigma(s), \bar{\boldsymbol{\xi}}_\sigma(t)) ds \\ &= (\eta_{u_t}, \xi_u) - (M(t, t) \bar{\boldsymbol{\xi}}_\sigma(t), \bar{\boldsymbol{\xi}}_\sigma(t)) + \int_0^t (M_s(t, s) \bar{\boldsymbol{\xi}}_\sigma(s), \bar{\boldsymbol{\xi}}_\sigma(t)) ds. \end{aligned}$$

Integrating from 0 to t , and using the Cauchy-Schwarz inequality and the bounds for M , we immediately obtain

$$\begin{aligned} \|\xi_u(t)\|^2 + \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma(t)\|^2 &\leq \|\xi_u(0)\|^2 + 2 \int_0^t \|\eta_{u_t}(s)\| \|\xi_u(s)\| ds \\ &\quad + C(a_0, a_1, T) \left(\int_0^t \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma(s)\|^2 ds + \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma(t)\| \int_0^t \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma(s)\| ds \right). \end{aligned}$$

Following the same arguments as in the proof of Theorem 4.2, we deduce that

$$\|\xi_u(t)\| + \|\alpha^{1/2} \bar{\boldsymbol{\xi}}_\sigma(t)\| \leq C \left(\|\xi_u(0)\| + \int_0^t \|\eta_{u_t}\| ds \right).$$

Finally, a use of the triangle inequality and Lemma 6.2 concludes the proof of theorem. \square

The discrete-in-time scheme for problem (1.10)-(1.11) is to seek $(U^n, \mathbf{Z}^n) \in W_h \times \mathbf{V}_h$, such that

$$(6.12) \quad \frac{2}{k}(\partial_t U^{1/2}, w_h) - (\nabla \cdot \mathbf{Z}^{1/2}, w_h) = \left(\frac{2}{k}u_1, w_h\right),$$

$$(6.13) \quad (\alpha \mathbf{Z}^{n+1/2}, \mathbf{v}_h) + (U^{n+1/2}, \nabla \cdot \mathbf{v}_h) + \epsilon^{n+1/2}(\mathcal{M}^{n+1/2}(\mathbf{Z}, \mathbf{v}_h)) = 0, \quad n \geq 0,$$

$$(6.14) \quad (\partial_t^2 U^n, w_h) - (\nabla \cdot \mathbf{Z}^{n+1/4}, w_h) = 0, \quad n \geq 1,$$

for all $(w_h, \mathbf{v}_h) \in W_h \times \mathbf{V}_h$, with given $(U^0, \mathbf{Z}^0) \in W_h \times \mathbf{V}_h$. In (6.13),

$$\epsilon^{n+1/2}(\mathcal{M}^{n+1/2}(\mathbf{Z}, \mathbf{v}_h)) = \frac{1}{2} (\epsilon^{n+1}(\mathcal{M}^{n+1}(\mathbf{Z}, \mathbf{v}_h)) + \epsilon^n(\mathcal{B}^n(\mathbf{Z}, \mathbf{v}_h))),$$

where

$$\epsilon^n(\mathcal{M}^n(\mathbf{Z}, \chi)) = k \sum_{j=0}^{n-1} (M(t_n, t_{j+1/2}) \mathbf{Z}^{j+1/2}, \chi).$$

This choice of the time discretization leads to a second order accuracy in k . Below we state the following theorem. Since the proof follows exactly the steps leading to Theorem 5.2 with appropriate changes, we skip the proof.

Theorem 6.2 *Let (u, σ) be the solution of (1.10)-(1.11) and $(U^n, \mathbf{Z}^n) \in W_h \times \mathbf{V}_h$ be the solution of (6.12)-(6.14). Assume that $U^0 = P_h u_0$ and $\mathbf{Z}^0 = P_h(A \nabla u_0)$. Then, there exists a positive constant C , independent of h and k , such that for small k with $k = O(h)$,*

$$(6.15) \quad \|u(t_{m+1}) - U^{m+1}\| \leq C(h^2 + k^2) (\|u_0\|_3 + \|u_1\|_2), \quad m = 0, 1, \dots, N-1.$$

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